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DELHOMME–LAFLAMME–POUZET–SAUER SPACE AS GROUPOID

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Abstract. Let $\mathbb{R}^+ = [0, \infty)$ and let d^+ be the ultrametric on $\mathbb{R}^+$ such that $d^+(x, y) = \max\{x, y\}$ for all different $x, y \in \mathbb{R}^+$. It is shown that the monomorphisms of the groupoid $(\mathbb{R}^+, d^+)$ coincide with the injective ultrametric-preserving functions and that the automorphisms of $(\mathbb{R}^+, d^+)$ coincide with the self-homeomorphisms of $\mathbb{R}^+$. The structure of endomorphisms of $(\mathbb{R}^+, d^+)$ is also described.

1. Introduction. Ultrametrics and pseudoultrametrics

Let denote by $\mathbb{R}^+$ the set of all nonnegative real numbers.

Definition 1.1. Let X be a nonempty set. An *ultrametric* on X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $(d(x, y) = 0) \Longleftrightarrow (x = y)$, the *positivity property*;
(ii) $d(x, y) = d(y, x)$, the *symmetry property*;
(iii) $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$, the *strong triangle inequality*.

If d is an ultrametric on X, then we will say that (X, d) is an *ultrametric space*.

The following ultrametric space was introduced by Delhomme, Laflamme, Pouzet and Sauer in [6] and this space is the main focus of our research.

Let us define a mapping $d^+ : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$d^+(p, q) := \begin{cases} 0, & \text{if} \quad p = q, \\ \max\{p, q\}, & \text{otherwise}. \end{cases} \tag{1.1}$$

Then d^+ is an ultrametric on $\mathbb{R}^+$.

Different properties of ultrametric spaces have been studied in [1, 6, 7, 12–23, 25–31, 33, 37, 38, 42, 43, 45, 46]. The importance of the space $(\mathbb{R}^+, d^+)$ and its subspaces in the theory of ultrametric spaces was noted by Yoshito Ishiki in [21].

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The useful generalization of the concept of ultrametric is the concept of pseudoultrametric.

Definition 1.2. Let X be a nonempty set and let $d : X \times X \to \mathbb{R}^+$ be a symmetric function such that $d(x, x) = 0$ holds for every $x \in X$. The function d is a *pseudoultrametric* on X if it satisfies the strong triangle inequality.

If d is a pseudoultrametric on X, then (X, d) is called a pseudoultrametric space.

Definition 1.3. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *ultrametric-preserving (pseudoultrametric-preserving)* if $(X, f \circ d)$ is an ultrametric space (pseudoultrametric space) for every ultrametric (pseudoultrametric) space (X, d).

Remark 1.4. Here we write $f \circ d$ for the mapping

$$X \times X \xrightarrow{d} \mathbb{R}^+ \xrightarrow{f} \mathbb{R}^+.$$

As in [11] we denote by $\mathbf{P_U}$ and the set of all ultrametric-preserving functions and, respectively, by $\mathbf{P_{PU}}$ the set of all pseudoultrametric-preserving ones. We also will use the following designations:

$\mathbf{In}(\mathbf{P_U})$ – the set of all injective $f \in \mathbf{P_U}$;

$\mathbf{In}(\mathbf{P_{PU}})$ – the set of all injective $f \in \mathbf{P_{PU}}$;

$\mathbf{Ai}(\mathbf{P_{PU}})$ – the set of all $f \in \mathbf{P_{PU}}$ with injective restriction $f_{|\mathbb{R}^+ \setminus f^{-1}(0)}$;

$\mathbf{ASI}$ – the set of all $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(0) = 0$ and strictly increasing restriction $f_{|\mathbb{R}^+ \setminus f^{-1}(0)}$;

$\mathbf{SI}$ – the set of all strictly increasing $f \in \mathbf{ASI}$;

$\mathbf{End}(S, *)$ – the set of all endomorphisms of a groupoid $(S, *)$;

$\mathbf{Mon}(S, *)$ – the set of all monomorphisms of $(S, *)$;

$\mathbf{Aut}(S, *)$ – the set of all automorphisms of $(S, *)$.

The main goal of the present paper is to prove the equalities

$$\mathbf{ASI} = \mathbf{End}(\mathbb{R}^+, d^+), \tag{1.2}$$

$$\mathbf{SI} = \mathbf{Mon}(\mathbb{R}^+, d^+). \tag{1.3}$$

The paper is organized as follows. The next section contains some definitions and results related to groupoids, ultrametric-preserving functions, and pseudoultrametric-preserving ones.

The main results of the paper are given in Section 3. Equalities (1.2) and (1.3) are proved in Theorems 3.6 and 3.9 respectively.

Theorem 3.14 shows that $\mathbf{Aut}(\mathbb{R}^+, d^+)$ coincides with the set of all self-homeomorphisms of $\mathbb{R}^+$.

2. Preliminaries on ultrametric-preserving functions and groupoids

Recall that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *increasing* iff

$$(a \geqslant b) \Longrightarrow (f(a) \geqslant f(b))$$

holds for all $a, b \in \mathbb{R}^+$. Moreover, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is *strictly increasing* iff

$$(a > b) \Longrightarrow (f(a) > f(b))$$

holds for all distinct $a, b \in \mathbb{R}^+$.

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *amenable* if $f^{-1}(0) = \{0\}$.

P. Ponsgriiam and I. Termwuttipong found the following simple characterization of $\mathbf{P_U}$ in [33].

Theorem 2.1. *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is ultrametric-preserving if and only if f is amenable and increasing.*

The next extension of Theorem 2.1 was obtained in [9].

Proposition 2.2. *The following conditions are equivalent for every function $f : \mathbb{R}^+ \to \mathbb{R}^+$.*

(i) *f is increasing and $f(0) = 0$ holds.*
(ii) *f is pseudoultrametric-preserving.*

Remark 2.3. The set $\mathbf{P_U}$ of ultrametric-preserving functions was also studied in [2, 3, 10, 24] and [44].

Proposition 2.2 and the definition of **SI** imply the following corollary.

Corollary 2.4. *The equality*

$$\mathbf{SI} = \mathbf{In}(\mathbf{P_{PU}}) \tag{2.1}$$

holds.

Proof. To prove (2.1) it is sufficient to note that an increasing function f is strictly increasing if and only if f is injective. □

Lemma 2.5. *Let us consider arbitrary $f \in \mathbf{In}(\mathbf{P_{PU}})$. Then f is ultrametric-preserving,*

$$f \in \mathbf{P_U}. \tag{2.2}$$

Proof. It follows from $f \in \mathbf{In}(\mathbf{P_{PU}})$ that $f \in \mathbf{P_{PU}}$. Since f belongs to $\mathbf{P_{PU}}$, f is increasing and the equality

$$f(0) = 0 \tag{2.3}$$

holds by Proposition 2.2. The injectivity of f and (2.3) imply

$$f^{-1}(0) = \{0\}.$$

Thus f is amenable and increasing. Hence (2.2) holds by Theorem 2.1 □

Lemma 2.5 gives us the next proposition wich will be used in Section 3 below.

Proposition 2.6. *The equality*

$$\mathbf{In}(\mathbf{P_{PU}}) = \mathbf{In}(\mathbf{P_U}) \tag{2.4}$$

holds.

Proof. Theorem 2.1 and Proposition 2.2 imply the inclusion

$$\mathbf{In}(\mathbf{P_{PU}}) \supseteq \mathbf{In}(\mathbf{P_U}). \tag{2.5}$$

Suppose now that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an arbitrary element of the set $\mathbf{In}(\mathbf{P_{PU}})$. Then the membership

$$f \in \mathbf{P_U} \tag{2.6}$$

is valid by Lemma 2.5. Since f is injective, (2.6) implies

$$f \in \mathbf{In}(\mathbf{P_U}).$$

Consequently the inclusion

$$\mathbf{In}(\mathbf{P_{PU}}) \subseteq \mathbf{In}(\mathbf{P_U})$$

holds. The last inclusion and (2.5) give us (2.4). □

Let us recall some definitions connected with the concept of groupoid.

Definition 2.7. A *groupoid* is a pair $(S, *)$ consisting of a set S and a binary operation $* : S \times S \to S$ which is called the *composition* on S. An element $e \in S$ of a groupoid $(X, *)$ is said to be the *identity* of S if the equalities

$$e * s = s * e = s$$

hold for every $s \in S$.

Remark 2.8. Nicolas Bourbaki uses the term "unital magma" to refer to a groupoid with identity element (see Definition 2 in [4, p. 12]).

It is easy to prove that, for arbitrary groupoid $(S, *)$, the identity element is unique if it exists, see, for example [41], p. 111, Proposition 1. In what follows we denote such element as 1_S.

Let us consider the basic example of a groupoid for us.

Example 2.9. Let $(\mathbb{R}^+, d^+)$ be the ultrametric space defined by formula (1.1). Then $(\mathbb{R}^+, d^+)$ is a groupoid with the composition d^+ and the identity element

$$1_{\mathbb{R}^+} = 0. \tag{2.7}$$

To see that (2.7) holds it suffices to note that the equalities

$$d^+(x, 0) = x = d^+(0, x)$$

follow from (1.1) for each $x \in \mathbb{R}^+$.

Definition 2.10. A groupoid $(S, *)$ is said to be a *monoid* if S contains an identity element and $*$ is associative.

Example 2.11. Let us define a composition $\vee$ on the set $\mathbb{R}^+$ as

$$x \vee y = \max\{x, y\}$$

for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, \vee)$ is a monoid with the identity element

$$1_{\mathbb{R}^+} = 0. \tag{2.8}$$

To see that $(\mathbb{R}^+, \vee)$ really is a monoid and (2.8) holds, it suffices to note that the equalities

$$\max\{\{x, y\}, z\} = \max\{x, y, z\} = \max\{x, \{y, z\}\}$$

and

$$\max\{x, 0\} = \max\{0, x\} = x$$

are satisfied for all $x, y, z \in \mathbb{R}^+$.

Proposition 2.12. *The groupoid $(\mathbb{R}^+, d^+)$ is not a monoid.*

Proof. Indeed, let $x, y \in \mathbb{R}^+$ and let the double inequality

$$x > y > 0 \tag{2.9}$$

hold. Then using (1.1) and (2.9) we obtain

$$d^+(d^+(x, x), y) = d^+(0, y) = y \neq 0 = d^+(x, x) = d^+(x, d^+(x, y)).$$

Thus the composition d^+ is not associative. □

Definition 2.13. Let $S = (S, *, 1_S)$ be a groupoid. Then a mapping $\Phi : S \to S$ is called an *endomorphism* if, for all $x, y \in S$, we have

$$\Phi(x * y) = \Phi(x) * \Phi(y)$$

and the equality

$$\Phi(1_S) = 1_S$$

holds. An injective endomorphism $S \to S$ is called a *monomorphism* of S. The bijective endomorphisms of S are called the *authomorphisms* of S.

Lemma 2.14. *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an endomorphism of the groupoid $(\mathbb{R}^+, d^+)$ if and only if $f(0) = 0$ and the equality*

$$f(d^+(x, y)) = d^+(f(x), f(y))$$

holds for all $x, y \in \mathbb{R}^+$.

Proof. It follows from Definitions 2.7 and 2.13. □

3. Main results

Let us turn now to the set $\mathbf{End}(\mathbb{R}^+, d^+)$ of endomorphisms of the groupoid $(\mathbb{R}^+, d^+)$.

Lemma 3.1. *The inclusion*

$$\mathbf{End}(\mathbb{R}^+, d^+) \subseteq \mathbf{P_{PU}} \tag{3.1}$$

holds.

Proof. Let us consider an arbitrary endomorphism f of the groupoid $(\mathbb{R}^+, d^+)$,

$$f \in \mathbf{End}(\mathbb{R}^+, d^+). \tag{3.2}$$

To prove inclusion (3.1) we must show that

$$f \in \mathbf{P_{PU}}. \tag{3.3}$$

By Proposition 2.2 membership (3.3) holds if f is increasing and satisfies the equality

$$f(0) = 0. \tag{3.4}$$

Equality (3.4) follows from Definition 2.13 and membership (3.2). Thus, it is enough to prove that f is an increasing function, i.e.

$$f(x) \leqslant f(y) \tag{3.5}$$

holds whenever $x, y \in \mathbb{R}^+$ and

$$x \leqslant y. \tag{3.6}$$

Let us consider arbitrary $x, y \in \mathbb{R}^+$ satysfying (3.6). By Lemma 2.14 membership (3.2) implies the equality

$$f(d^+(x, y)) = d^+(f(x), f(y)). \tag{3.7}$$

Using (1.1) and (3.6) we obtain the equality $d^+(x, y) = y$. The last equality and (3.7) give us

$$f(y) = d^+(f(x), f(y)). \tag{3.8}$$

Suppose first that $f(y) = 0$. Then we can rewrite (3.8) us

$$d^+(f(x), f(y)) = 0. \tag{3.9}$$

Since d^+ is an ultrametric, (3.9) is valid iff

$$f(x) = f(y),$$

that implies (3.5). If $f(y) \neq 0$, then (3.8) implies the equality

$$f(y) = \max\{f(x), f(y)\}. \tag{3.10}$$

Inequality (3.5) follows from (1.1) and (3.10).

The proof is completed. □

The following example shows that $\mathbf{End}(\mathbb{R}^+, d^+)$ is a proper subset of $\mathbf{P_{PU}}$.

Example 3.2. Let a be an arbitrary point of $(0, \infty)$. Let us define $f : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f(t) = \begin{cases} 0, & \text{if} \quad t = 0, \\ a, & \text{otherwise.} \end{cases}$$

Proposition 2.2 implies $f \in \mathbf{P_{PU}}$. Suppose that f is an endomorphism of $(\mathbb{R}^+, d^+)$. Then, by Lemma 2.14, the equality

$$f(d^+(x, y)) = d^+(f(x), f(y)) \tag{3.11}$$

holds for all $x, y \in \mathbb{R}^+$. In particular, for $x = 0$ and $y = 1$, using (1.1) we obtain

$$f(a) = f(d^+(0, 1)) = d^+(f(0), f(1)) = d^+(0, a) = a. \tag{3.12}$$

Similarly if $x = 1$ and $y = 2$ then (3.11) implies

$$f(a) = f^+(d^+(1, 2)) = d^+(f(1), f(2)) = d^+(a, a) = 0. \tag{3.13}$$

It follows from (3.12) and (3.13) that $a = 0$, contrary to $a > 0$. Thus we have

$$f \in \mathbf{P_{PU}} \quad \text{and} \quad f \notin \mathbf{End}(\mathbb{R}^+, d^+).$$

By analysing Example 3.2, we can prove the following lemma.

Lemma 3.3. *Let $a \in (0, \infty)$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $f(0) = 0$ and*

$$f(x) = f(y) = a$$

for some different $x, y \in (0, \infty)$. Then f is not an endomorphism of the groupoid $(\mathbb{R}^+, d^+)$,

$$f \notin \mathbf{End}(\mathbb{R}^+, d^+).$$

Lemma 3.4. *The inclusion*

$$\mathbf{End}(\mathbb{R}^+, d^+) \subseteq \mathbf{ASI} \tag{3.14}$$

holds.

Proof. Let us consider an arbitrary function

$$f \in \mathbf{End}(\mathbb{R}^+, d^+). \tag{3.15}$$

We must show that

$$f \in \mathbf{ASI}. \tag{3.16}$$

Suppose contrary that

$$f \notin \mathbf{ASI}. \tag{3.17}$$

By Lemma 3.1, membership (3.15) implies

$$f \in \mathbf{P}_{\mathbf{PU}}. \tag{3.18}$$

Now using (3.18) and Proposition 2.2 we obtain that f is increasing and satisfies $f(0) = 0$. Hence (3.17) implies that the restriction of f on the set $\mathbb{R}^+ \setminus f^{-1}(0)$ is not strictly increasing. An increasing function is strictly increasing iff it is an injective function. Thus $f_{|\mathbb{R}^+ \setminus f^{-1}(0)}$ is not injective. Consequently there are $a \in (0, \infty)$ and $x, y \in \mathbb{R}^+ \setminus f^{-1}(0)$ such that

$$f(x) = f(y) = a. \tag{3.19}$$

The equality $f(0) = 0$ implies

$$\mathbb{R}^+ \setminus f^{-1}(0) \subseteq (0, \infty).$$

Consequently the points a, x and y belong to $(0, \infty)$. Now applying Lemma 3.3 we obtain

$$f \notin \mathbf{End}(\mathbb{R}^+, d^+)$$

contrary to (3.15). Thus (3.16) is valid, that implies (3.14). The proof is completed. □

Lemma 3.5. *The equality*

$$\mathbf{ASI} = \mathbf{Ai}(\mathbf{P}_{\mathbf{PU}})$$

holds.

Proof. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $f(0) = 0$. Then the restriction $f_{|\mathbb{R}^+ \setminus f^{-1}(0)}$ is strictly increasing iff this restriction is injective. □

The following theorem is the first main result of the paper.

Theorem 3.6. *The equalities*

$$\mathbf{End}(\mathbb{R}^+, d^+) = \mathbf{ASI} = \mathbf{Ai}(\mathbf{P}_{\mathbf{PU}}) \tag{3.20}$$

holds.

Proof. By Lemma 3.5 we have the equality

$$\mathbf{ASI} = \mathbf{Ai}(\mathbf{P}_{\mathbf{PU}}).$$

Moreover, Lemma 3.4 gives us the inclusion

$$\mathbf{End}(\mathbb{R}^+, d^+) \subseteq \mathbf{ASI}.$$

Hence (3.20) holds iff

$$\mathbf{ASI} \subseteq \mathbf{End}(\mathbb{R}^+, d^+). \tag{3.21}$$

To prove (3.21) let us consider an arbitrary function $f \in \mathbf{ASI}$. It is sufficient to show that

$$f \in \mathbf{End}(\mathbb{R}^+, d^+). \tag{3.22}$$

By Lemma 2.14 membership (3.22) is valid if

$$f(d^+(x,y)) = d^+(f(x), f(y)) \tag{3.23}$$

holds for all $x, y \in \mathbb{R}^+$.

Let us consider arbitrary $x, y \in \mathbb{R}^+$. Suppose first that

$$x, y \in f^{-1}(0). \tag{3.24}$$

Assume, without loss of generality, that

$$x \geqslant y.$$

Then $d^+(x,y) = x$ holds by (1.1) and we have

$$f(x) = f(y) = 0 \tag{3.25}$$

by (3.24).

Now using (3.24) and (3.25) we obtain (3.22),

$$f(d^+(x,y)) = f(x) = 0 = d^+(0,0) = d^+(f(x), f(y)).$$

Suppose now that exactly one from the points x, y belongs to $f^{-1}(0)$. WLOG , let $x \in \mathbb{R}^+ \setminus f^{-1}(0)$ and $y \in f^{-1}(0)$. Then we have

$$f(x) > 0 \quad \text{and} \quad f^{-1}(y) = 0. \tag{3.26}$$

The membership $f \in \mathbf{ASI}$ implies that f is increasing. Consequently (3.26) implies the inequality

$$d^+(f(x), f(y)) = f(x). \tag{3.27}$$

Now using (3.26), (3.27) and (1.1) we obtain

$$f(d^+(x,y)) = f(x) \tag{3.28}$$

and

$$d^+(f^+(x), f^+(y)) = f(y). \tag{3.29}$$

Thus (3.23) holds. To complete the proof of (3.22), it remains to consider the case when

$$x, y \in \mathbb{R}^+ \setminus f^{-1}(0). \tag{3.30}$$

If $x = y$, then $f(x) = f(y)$ and consequently we have

$$f(d^+(x,y)) = f(0) = 0 = d^+(f(x), f(y)),$$

that implies (3.22). Suppose that $x \neq y$. WLOG let $x > 0$.

Then (3.30) implies

$$f(x) > f(y).$$

Therefore (3.28) and (3.29) follows as above.

Thus equality (3.23) is valid for all $x, y \in \mathbb{R}^+$, that implies the validity of (3.22).

The proof is completed. □

The next theorem was proved in [11].

Theorem 3.7. *The sets* $\mathbf{End}(\mathbb{R}^+, \vee)$ *and* $\mathbf{P_{PU}}$ *are the same,*

$$\mathbf{End}(\mathbb{R}^+, \vee) = \mathbf{P_{PU}}. \tag{3.31}$$

This theorem implies the following

Lemma 3.8. *The equality*

$$\mathbf{Mon}(\mathbb{R}^+, \vee) = \mathbf{In}(\mathbf{P_{PU}}) \tag{3.32}$$

holds.

Proof. Equality (3.32) follows from (3.31) and the definitions of $\mathbf{Mon}(\mathbb{R}^+, \vee)$ and $\mathbf{In}(\mathbf{P_{PU}})$. □

The next theorem is the second main result of the paper.

Theorem 3.9. *The equalities*

$$\mathbf{Mon}(\mathbb{R}^+, d^+) = \mathbf{SI} = \mathbf{In}(\mathbf{P_U}) = \mathbf{In}(\mathbf{P_{PU}}) = \mathbf{Mon}(\mathbb{R}^+, \vee) \tag{3.33}$$

hold.

Proof. The equality

$$\mathbf{Mon}(\mathbb{R}^+, \vee) = \mathbf{In}(\mathbf{P_{PU}})$$

was proved in Lemma 3.8. Lemma 2.5 and Proposition 2.6 give us the equalities

$$\mathbf{SI} = \mathbf{In}(\mathbf{P_U}) = \mathbf{In}(\mathbf{P_{PU}}).$$

Consequently (3.33) holds iff

$$\mathbf{Mon}(\mathbb{R}^+, d^+) = \mathbf{SI}. \tag{3.34}$$

Theorem 3.6 implies the equality

$$\mathbf{End}(\mathbb{R}^+, d^+) = \mathbf{ASI}.$$

The last equality and Definition 2.13 give us

$$\mathbf{Mon}(\mathbb{R}^+, d^+) = \{f \in \mathbf{ASI} : f \text{ is injective}\}. \tag{3.35}$$

Now using the definition of the sets **ASI** and **SI** we see that

$$\mathbf{SI} = \{f \in \mathbf{ASI} : f \text{ is injective}\}. \tag{3.36}$$

Equality (3.34) follows from equalities (3.35)–(3.36).

The proof is completed. □

Corollary 3.10. *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monomorphism of the groupoid $(\mathbb{R}^+, d^+)$ if and only if f is amenable and strictly increasing.*

Proof. It follows from (3.33) and Theorem 2.1. □

Our next goal is to characterize the set $\mathbf{Aut}(\mathbb{R}^+, d^+)$ of all automorphisms of the groupoid $(\mathbb{R}^+, d^+)$.

Lemma 3.11. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing. Write*

$$f(x_0 + 0) := \inf\{f(x) : x \in (x_0, \infty)\},$$

and

$$f(x_0 - 0) := \sup\{f(x) : x \in [0, x_0)\}$$

for each $x_0 \in (0, \infty)$ and, in addition, denote by $f(0 + 0)$ the infinum of the set $\{f(x) : x \in (0, \infty)\}$,

$$f(0 + 0) := \inf\{f(x) : x \in (0, \infty)\}.$$

Then f is a continuous function on $\mathbb{R}^+$ if and only if

$$f(0) = f(0 + 0),$$

and the equalities

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0)$$

hold for each $x_0 \in (0, \infty)$.

For a proof see [32, pages 204–205].

The next lemma follows directly from Theorem 5 of [5] (see pages 338–339).

Lemma 3.12. *A mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a self-homeomorphism of $\mathbb{R}^+$ if and only if*

$$f(\mathbb{R}^+) = \mathbb{R}^+$$

and f is strictly monotonic and continuous on $\mathbb{R}^+$.

Remark 3.13. In Lemma 3.12 and Theorem 3.14 we consider $\mathbb{R}^+$ as a topological space with topology included by the usual metric

$$d(x, y) = |x - y|, \quad \text{for } x, y \in \mathbb{R}^+.$$

Theorem 3.14. *The following statements are equivalent for every function $F : \mathbb{R}^+ \to \mathbb{R}^+$.*

(i) *$F \in \mathbf{Aut}(\mathbb{R}^+, d^+)$.*

(ii) *F is strictly increasing and satisfies the equality*

$$F(\mathbb{R}^+) = \mathbb{R}^+. \tag{3.37}$$

(iii) *F is a self-homeomorphism of $\mathbb{R}^+$.*

Proof. (*i*) $\Longrightarrow$ (*ii*). Let F belong to $\mathbf{Aut}(\mathbb{R}^+, d^+)$. Then

$$F \in \mathbf{Mon}(\mathbb{R}^+, d^+)$$

holds and, consequently, F is strictly increasing by Corollary 3.10. The validity of (3.37) follows from Definition 2.13. Thus (*ii*) holds.

(*ii*) $\Longrightarrow$ (*iii*). Let (*ii*) hold. Since F is strictly increasing, F is strictly monotonic. Moreover, (3.37) holds by statement (*ii*). Thus, by Lemma 3.12, statement (*iii*) is valid if F is continuous.

Suppose contrary that F is a discontinuous function. Then, by Lemma 3.11,

$$f(0) < f(0+0) \tag{3.38}$$

or there is $x_0 \in (0, \infty)$ such that

$$f(x_0 - 0) < f(x_0 + 0). \tag{3.39}$$

If (3.38) holds, then using (3.11) we obtain the equality

$$(f(0), f(0+0)) \cap f(\mathbb{R}^+) = \emptyset \tag{3.40}$$

where

$$(f(0), f(0+0)) = \{x \in \mathbb{R}^+ : f(0) < x < f(0+0)\}. \tag{3.41}$$

The interval $(f(0), f(0+0))$ is nonempty by (3.38). Hence $F(\mathbb{R}^+)$ is a proper subset of $\mathbb{R}^+$,

$$F(\mathbb{R}^+) \subsetneq \mathbb{R}^+ \tag{3.42}$$

contrary to (3.37).

If inequality (3.39) satisfied for some point $x_0 \in (0, \infty)$, then reasoning in a similar way we can prove that (3.42) also holds. Since (3.42) contradicts (3.37), statement (*iii*) follows.

(*iii*) $\Longrightarrow$ (*i*). Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a homeomorphism. We must prove that

$$F \in \mathbf{Aut}(\mathbb{R}^+, d^+). \tag{3.43}$$

By Lemma 3.12, equality (3.37) holds. Consequently, by Definition 2.13, (3.43) is valid iff

$$F \in \mathbf{Mon}(\mathbb{R}^+, d^+). \tag{3.44}$$

It follows from Corollary 3.10 that (3.44) holds if F is strictly increasing and satisfies the equality

$$F(0) = 0. \tag{3.45}$$

Let us prove equality (3.45).

The restriction of F on the interval $(0, \infty)$ is a homeomorphism of $(0, \infty)$ on $\mathbb{R}^+ \setminus \{F(0)\}$. Consequently $\mathbb{R}^+ \setminus \{F(0)\}$ is a connected subset of $\mathbb{R}^+$. The last statement is valid iff (3.45) holds.

Equality (3.45) and Lemma 3.12 imply that F is strictly increasing. Indeed, let x_0 be an arbitrary point of $(0, \infty)$. Since F is a bijective mapping (3.45) implies that

$$F(x_0) \in (0, \infty).$$

Thus the inequality

$$F(x_0) > F(0) \tag{3.46}$$

holds. By Lemma 3.12, the mapping F is strictly monotonic. The last property and (3.46) imply that F is strictly increasing.

Thus (3.44) holds. Statement (i) follows.

The proof is completed. □

It should be noted that the concept of ultrametric-preserving functions has been extended to the special case of "ultrametric distances" (see [8]). These distances were introduced by Priess–Crampe and Ribenboim [34] and were studies by different researchers [35, 36, 39, 40]. It seems to be interesting to find generalizations of the main results of the present paper to groupoids generated by such distances.

Conflict of interest statement

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References

[1] Beyrer, J., and Schroeder, V. Trees and ultrametric möbius structures. *p-adic Numbers Ultrametr. Anal. Appl. 9*, 4 (2017), 247–256.

[2] Bilet, V., and Dovgoshey, O. On monoid of metric preserving functions. *Front. Appl. Math. Stat. 10:1420671* (2024).

[3] Bilet, V., Dovgoshey, O., and Shanin, R. Ultrametric preserving functions and weak similarities of ultrametric spaces. *p-adic Numbers Ultrametr. Anal. Appl. 13*, 3 (2021), 186–203.

[4] Bourbaki, N. *Algebra I. Chapters 1-3.* Addison-Wesley Publishing Company, 1974.

[5] Bourbaki, N. *General Topology. Chapters 1-4*, vol. 18 of *Elements of Mathematics.* Springer-Verlag, Berlin, Heidelberg, 1995.

[6] Delhommé, C., Laflamme, C., Pouzet, M., and Sauer, N. Indivisible ultrametric spaces. *Topology and its Applications 155*, 14 (2008), 1462–1478.
[7] Dovgoshey, O. Combinatorial properties of ultrametrics and generalized ultrametrics. *Bull. Belg. Math. Soc. Simon Stevin 27*, 3 (2020), 379–417.
[8] Dovgoshey, O. Combinatorial properties of ultrametrics and generalized ultrametrics. *Bull. Belg. Math. Soc. Simon Stevin 27*, 3 (2020), 379–417.
[9] Dovgoshey, O. On ultrametric-preserving functions. *Math. Slovaca 70*, 1 (2020), 173–182.
[10] Dovgoshey, O. Strongly ultrametric preserving functions. *Topology and its Applications 351* (2024), 108931.
[11] Dovgoshey, O. Ultrametric-preserving functions as monoid endomorphisms. *arXiv:2406.07166v2* (2024), 1–17.
[12] Dovgoshey, O., and Dordovskyi, D. Ultrametricity and metric betweenness in tangent spaces to metric spaces. *p-adic Numbers Ultrametr. Anal. Appl. 2*, 2 (2010), 100–113.
[13] Dovgoshey, O., and Kostikov, A. Locally finite ultrametric spaces and labeled trees. *Journal of Mathematical Sciences 276*, 5 (2023), 614–637.
[14] Dovgoshey, O., and Martio, O. Blow up of balls and coverings in metric spaces. *Manuscripta Math. 127* (2008), 89–120.
[15] Dovgoshey, O., and Martio, O. Products of metric spaces, covering numbers, packing numbers and characterizations of ultrametric spaces. *Rev. Roumaine Math. Pures. Appl. 54*, 5-6 (2009), 423–439.
[16] Dovgoshey, O., and Petrov, E. Subdominant pseudoultrametric on graphs. *Sb. Math 204*, 8 (2013), 1131–1151.
[17] Gomory, R. E., and Hu, T. C. Multi-terminal network flows. *SIAM 9*, 4 (1961), 551–570.
[18] Groot, J. d. Non-Archimedean metrics in topology. *Proc. Amer. Math. Soc. 7*, 5 (1956), 948–953.
[19] Ibragimov, Z. Möbius maps between ultrametric spaces are local similarities. *Ann. Acad. Sci. Fenn. Math. 37* (2012), 309–317.
[20] Ishiki, Y. An embedding, an extension, and an interpolation of ultrametrics. *p-Adic Numbers Ultrametric Anal. Appl. 13*, 2 (2021), 117–147.
[21] Ishiki, Y. Constructions of Urysohn universal ultrametric spaces. *p-Adic Numbers Ultrametric Anal. Appl. 15*, 4 (2023), 266–283.
[22] Ishiki, Y. Simultaneous extensions of metrics and ultrametrics of high power. *Topology Appl. 336* (2023), 37. Id/No 108624.
[23] Ishiki, Y. Uniqueness and homogeneity of non-separable Urysohn universal ultrametric spaces. *Topology Appl. 342* (2024), 11. Id/No 108762.
[24] Khemaratchatakumthorn, T., Pongsriiam, P., and Samphavat, S. Further remarks on b-metrics, metric-preserving functions, and other related metrics. *Int. J. Math. Comput. Sci. 14*, 2 (2019), 473–480.
[25] Kirk, W. A., and Shahzad, N. Some fixed point results in ultrametric spaces. *Topology Appl. 159* (2012), 3327–3334.
[26] Lemin, A. J. On isosceles metric spaces. *Functional Analysis and its Applications* (1984), 26–31. (in Russian).
[27] Lemin, A. J. On the stability of the property of a space being isosceles. *Russ. Math. Surveys 39*, 5 (1984), 283–284.

[28] Lemin, A. J. Proximity on isosceles spaces. *Russ. Math. Surveys 39*, 1 (1984), 169–170.

[29] Lemin, A. J. Isometric embedding of isosceles (non-Archimedean) spaces in Euclidean spaces. *Soviet Math. Dokl. 32*, 3 (1985), 740–744.

[30] Lemin, A. J. An application of the theory of isosceles (ultrametric) spaces to the Trnkova-Vinarek theorem. *Comment. Math. Univ. Carolinae 29*, 3 (1988), 427–434.

[31] Lemin, A. J. The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, real graduated lattices $\mathbf{LAT}^*$. *Algebra Universalis 50*, 1 (2003), 35–49.

[32] Natanson, I. P. *Theory of Functions of a Real Variable*, vol. I. Frederick Ungar Publishing Ko, New-Yourk, 1983.

[33] Pongsriiam, P., and Termwuttipong, I. Remarks on ultrametrics and metric-preserving functions. *Abstr. Appl. Anal. 2014* (2014), 1–9.

[34] Priess-Crampe, S., and Ribenboim, P. Fixed points, combs and generalized power series. *Abh. Math. Sem. Univ. Hamburg 63* (1993), 227–244.

[35] Priess-Crampe, S., and Ribenboim, P. Generalized ultrametric spaces I. *Abh. Math. Sem. Univ. Hamburg 66* (1996), 55–73.

[36] Priess-Crampe, S., and Ribenboim, P. Generalized ultrametric spaces II. *Abh. Math. Sem. Univ. Hamburg 67* (1997), 19–31.

[37] Qiu, D. Geometry of non-Archimedian Gromov–Hausdorff distance. *p-adic Numbers Ultrametr. Anal. Appl. 1*, 4 (2009), 317–337.

[38] Qiu, D. The structures of Hausdorff metric in non-Archimedian spaces. *p-adic Numbers Ultrametr. Anal. Appl. 6*, 1 (2014), 33–53.

[39] Ribenboim, P. The new theory of ultrametric spaces. *Periodica Math. Hung. 32*, 1–2 (1996), 103–111.

[40] Ribenboim, P. The immersion of ultrametric spaces into Hahn Spaces. *J. of Algebra 323*, 5 (2009), 1482–1493.

[41] Rosenfeld, A. *An introduction to algebraic structures.* Holden–Day, San Francisco-Cambridge-London-Amsterdam, 1968.

[42] Samphavat, S., Khemaratchatakumthorn, T., and Pongsriiam, P. Remarks on b-metrics, ultrametrics, and metric-preserving functions. *Mathematica Slovaca 70*, 1 (2020), 61–70.

[43] Samphavat, S., and Prinyasart, T. On ultrametrics, b-metrics, w-distances, metric-preserving functions, and fixed point theorems. *Fixed Point Theory Algorithms Sci. Eng.*, 9 (2024).

[44] Vallin, R. W., and Dovgoshey, O. A. P-adic metric preserving functions and their analogues. *Math. Slovaca 71*, 2 (2021), 391–408.

[45] Vaughan, J. E. Universal ultrametric spaces of smallest weight. *Topology Proc. 24* (1999), 611–619.

[46] Vestfrid, I. On the universal ultrametric space. *Ukrainin Math. J. 46*, 12 (1994), 1890–1898.

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