Solving a Non-linear Fractional Convection-Diffusion Equation Using Local Discontinuous Galerkin method

Hamid Safdari[a], Majid Rajabzadeh[a,*], Moein Khalighi[b]

[a] *Department of Applied Mathematics, Faculty of Mathematical sciences, Shahid Rajaee Teacher Training University, Tehran, Iran*
[b] *Department of computing, Faculty of Technology, University of Turku, Finland*

Abstract

We propose a local discontinuous Galerkin method for solving a nonlinear convection-diffusion equation consisting of a fractional diffusion described by a fractional Laplacian operator of order $0 < p < 2$, a nonlinear diffusion, and a nonlinear convection term. The algorithm is developed by the local discontinuous Galerkin method using Spline interpolations to achieve higher accuracy. In this method, we convert the main problem to a first-order system and approximate the outcome by the Galerkin method. In this study, in contrast to the direct Galerkin method using Legender polynomials, we demonstrate that the proposed method can be suitable for the general fractional convection-diffusion problem, remarkably improve stability and provide convergence order $O(h^{k+1})$, when k indicates the degree of polynomials. Numerical results have illustrated the accuracy of this scheme and compare it for different conditions.

Introduction

Experimental data continues to inspire and challenge mathematical modeling in various domains. In this regard, many scholars are inclined to introduce or develop more differential equations as new tools to describe complex phenomena. Particularly, in a physical system, some quantities–e.g. energy or particles–transferred due to two processes "diffusion and convection" can be represented by the combination of the diffusion and convection equations [1]. However, in the real world, there are many exceptions so that a process sometimes cannot be satisfied by classic deterministic models. Hence, incorporating more flexible mathematical tools such as fractional calculus into the former models could deal with some shortcomings and provide a suitable description of those special cases.

Fractional calculus is an active field in mathematics with impressive applications in a large range of science and engineering [2, 3, 4] that brings more degrees of freedom for differentiation in the modeling of various phenomena, such as complex networks [5, 6, 7], optimal control problems [8, 9], quantum mechanics and field theory [10, 11], and Viscoelastic systems [12]. Such flexible differential operators do not have unique definitions, however. In the literature, the Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions are examples of commonly used approaches [13], and one can find generalizations of fractional-order differential operators like Atangana–Balenau,

*Corresponding author
Email addresses: Hsafdari@sru.ac.ir (Hamid Safdari), majid.rajabzade@yahoo.com (Majid Rajabzadeh), moein.khalighi@utu.fi (Moein Khalighi)

Caputo–Fabrizio, Hadamard, conformable [14, 13], and new operators so-called Ψ-fractional derivative [15], Ψ-Hilfer [16, 17, 18], and Hilfer–Katugampola [19] have been proposed concerning the monotonic increasing function and also substituting the exponential function by the truncated Mittag-Leffler function.

In this study, we work on the fractional Laplacian, as generalized operator of fractional spatial derivatives, which can be defined by a singular integral [20, 21]

$$-(-\Delta)^{\frac{p}{2}}(\chi(x,t)) = c_p \int_{|z|>0} \frac{\chi(x+z,t)-\chi(x,t)}{|z|^{1+p}} dz, \quad p \in (0,2)\,,\ c_p > 0, \tag{1}$$

to solve a class of convection-diffusion equations described as follow:

$$\begin{aligned} &\tfrac{\partial \chi(x,t)}{\partial t} + \tfrac{\partial f(\chi)}{\partial x} = \tfrac{\partial}{\partial x}\Big(a\,(\chi)\,\tfrac{\partial \chi(x,t)}{\partial x}\Big) + b\Big(-(-\Delta)^{\frac{p}{2}}\Big)\chi(x,t), \quad (x,t) \in \mathbb{R}\times(0,T), \\ &\chi(x,0) = \chi_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2}$$

in which three terms $\frac{\partial f(\chi)}{\partial x}$, $\frac{\partial}{\partial x}\Big(a\,(\chi)\,\frac{\partial \chi(x,t)}{\partial x}\Big)$, and $\Big(-(-\Delta)^{\frac{p}{2}}\Big)$ are, respectively, the nonlinear convection, the nonlinear diffusion, and the fractional diffusion such that $f, a : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions, where $a \geq 0$ is bounded and $b \geq 0$ is constant. Let us assume $f(0) = 0$ in this paper, and the initial value χ_0 chosen in different spaces which depends on whether the equations are linear or nonlinear. Equation (2) with the nonlocal operator can have wide potential applications in representing molecular biology, dislocation dynamics, hydrodynamics [22], option pricing models for mathematical finance [23], explosives and semiconductors devices [24], and many other areas of research [25, 26, 27].

Numerical solutions of partial differential equations with fractional operator are not straightforward as much as counterparts including integer operator. Methods based on finite difference [28], boundary element [29], and finite element methods [30] have been proposed and successfully applied for solutions of fractional differential equations. However, a few numerical methods for solving fractional diffusion problems with fractional Laplacian operators have been developed (see e.g. [31]). For the first time, Reed and Hill [32] have introduced a class of finite element methods which have been led to different type of Galerkin methods such as discontinuous Galerkin (DG) scheme for the problems with less smoothness. The DG method have been applied for solving fractional convection-diffusion equations in [33, 34, 35]. Likewise, local discontinuous Galerkin (LDG) methods [34] have been appropriately exploited for time-dependent partial differential equations with higher derivatives by rewriting the equation into a first order system to able carry out the DG method to the system.

By considering the nonlinear diffusion term in (2), we investigate a generalization form of the proposed model by Xu and Hesthaven [36] in which the applied numerical solution was DG method based on Legendre polynomials. This method is directly applied after a suitable numerical flux has been derived for the diffusion term. Notwithstanding, it is not practical for convection-diffusion equations with a nonlinear diffusion term. Hence, we develop the numerical algorithm using LDG method based on Spline polynomials to solve the generalized fractional convection-diffusion equation (2) with the nonlinear diffusion term and fractional Laplacian order $p \in (0,2)$. To do so, we convert the equation into a system with lower order equations so that it allows us to apply the DG method and prove the stability estimates along with convergence results.

This paper is assembled as follows. In Section 1, we introduce some necessarily definitions and state fundamental lemmas. In Section 2, we will derive the LDG method for the fractional convection-diffusion equation. In sections 3 and 4, we discuss the stability and convergence of the fractional convection-diffusion equations. In Section 5 some numerical examples are provided to validate the proposed method.

1. Preliminaries

In this section, based on [37, 38], some required definitions in fractional calculus are stated. Left and right Riemann-Liouville fractional integral of order p are defined as

$$ {}_aI_x^p\, v(x) = \frac{1}{\Gamma(p)} \int_a^x (x-\theta)^{p-1} v(\theta) d\theta, \quad x > a, \quad p \in \mathbb{R}^+, \tag{3} $$

$$ {}_xI_a^p\, v(x) = \frac{1}{\Gamma(p)} \int_x^a (\theta - x)^{p-1} v(\theta) d\theta, \quad x < a, \quad p \in \mathbb{R}^+, \tag{4} $$

where $a \in \mathbb{R}$. For $p \in [n-1, n)$, the left and right Riemann-Liouville fractional derivative of order p are defined as follow

$$ {}_{-\infty}D_x^p v(x) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-\theta)^{n-p-1} v(\theta) d\theta, $$

$$ {}_xD_\infty^p v(x) = \frac{1}{\Gamma(n-p)} \left(-\frac{d}{dx}\right)^n \int_x^\infty (\theta - x)^{n-p-1} v(\theta) d\theta. $$

Definition 1. *For $0 < p < 1$ we define*

$$ \Delta_{-\frac{p}{2}} u(x) = -\frac{{}_{-\infty}I_x^p\, \chi(x) + {}_xI_\infty^p\, \chi(x)}{2\cos(p\pi/2)}. $$

When $1 < p < 2$, we have

$$ -(-\Delta)^{\frac{p}{2}} \chi(x) = \frac{d^2}{dx^2}\left(\Delta_{\frac{p-2}{2}} \chi\right) = \Delta_{\frac{p-2}{2}} \left(\frac{d^2\chi}{dx^2}\right) = \frac{d}{dx}\left(\Delta_{\frac{p-2}{2}} \frac{d\chi}{dx}\right). \tag{5} $$

Lemma 2. *[36] Suppose $\chi(x)$ is smoothness function on domain $\Omega \subset \mathbb{R}$. Ω_h is a discretization of the domain with length h such that χ is approximated by $\chi_h(x)$ in $\mathcal{P}_h^k$, where $\mathcal{P}_h^k$ denotes the set of all polynomials of degree less than or equal $k(\geq 1)$ interval width h. If $\chi_h(x)$ in each mesh domain I_r is a polynomial of degree up to order k, and $(\chi, v)_{I_r} = (\chi_h, v)_{I_r}, \forall v \in \mathcal{P}^k$, then for $-1 < p \leq 0$ we have*

$$ \left\| \Delta_{p/2} \chi(x) - \Delta_{p/2} \chi_h(x) \right\|_{L_2} \leq C h^{k+1}, $$

and for $k \geq n, 0 \leq n-1 < p \leq n$ we have

$$ \left\| -(-\Delta)^{p/2} \chi(x) + (-\Delta)^{p/2} \chi_h(x) \right\|_{L_2} \leq C h^{k+1-n}, $$

where C is a constant independent of h.

Considering that the problem is defined in $\mathbb{R}$, for numerical implementation, we define a finite domain $\Omega = [a,b] \subset \mathbb{R}$ such that χ has a solution of the problem to the domain Ω. Assuming the $N+1$ discretization $a = x_1 < x_2 < x_3 < ... < x_{N+1} = b$,, we provide a mesh by this notation $I_r = (x_r, x_{r+1}),\ \Delta x_r = x_{r+1} - x_r$. We define the piece wise polynomial space V^k on the mesh as,

$$V^k = \left\{ v : \Omega \to \mathbb{R} \mid v|_{I_r} \in \mathcal{P}^k(I_r),\ r = 1,2,3,...,N \right\}.$$

Suppose Spline polynomials are indicated by $\{\zeta_{0,r}, \zeta_{1,r}, ..., \zeta_{k,r}\}$, where $\zeta_{j,r} \in \mathcal{P}^j(I_r)$. Thus, we can convert each function in $\mathcal{P}^k(I_r)$ to a linear combination of these polynomials.

2. The LDG schemes

As $1 < p < 2$ and from Lemma 2, the order of approximation of the fractional laplacian operator is $k+1-n$. To develop DG scheme for the fractional derivative with a high order, the fractional derivative can be rewritten as a combination of first order derivatives and a fractional integral then convert the generated system of equations with a lower order. By considering (5), introducing three variables s, q, d, and defining $a(\chi)\chi_x = \left(\sqrt{a(\chi)}\right) g(\chi)_x$ where $g(\chi) = \int^{\chi} \sqrt{a}\, \mathrm{d}x$, equation (2) can be converted into a system of equations as follow:

$$\begin{aligned}
&\chi_t + \left(f(\chi) - \sqrt{a(\chi)}q\right)_x = \sqrt{b}\tfrac{\partial}{\partial x}s,\\
&q - g(\chi)_x = 0,\\
&s = \Delta_{\frac{p-2}{2}} d,\\
&d = \sqrt{b}\tfrac{\partial}{\partial x}\chi.
\end{aligned}$$

We seek (χ, q, s, d) as an approximation of $(\chi_h, q_h, s_h, d_h) \in V_h$ so that for any $v, w, z, l \in V^k$, we have

$$\begin{aligned}
&\left(\tfrac{\partial \chi_h(x,t)}{\partial t}, v(x)\right)_{I_r} + \left(\left(f(\chi_h) - \sqrt{a(\chi_h)}q_h\right)_x, \tfrac{\partial v}{\partial x}\right)_{I_r} = \sqrt{b}\left(\tfrac{\partial s_h}{\partial x}, v\right)_{I_r},\\
&(q_h, w(x))_{I_r} = (g(\chi_h)_x, w(x))_{I_r},\\
&(s_h, z(x))_{I_r} = \left(\Delta_{\frac{p-2}{2}} d_h, z(x)\right)_{I_r},\\
&(d_h, l(x))_{I_r} = b\left(\tfrac{\partial \chi_h}{\partial x}, l(x)\right)_{I_r},\\
&(\chi_h(x,0), v(x))_{I_r} = (\chi_0(x), v(x))_{I_r}.
\end{aligned} \tag{6}$$

Where such a notation $(\chi, v)_I = \int_I \chi(x)v(x)\mathrm{d}x$ denotes the standard inner product over the corresponding elements. By considering the following notation

$$\chi^{\pm}(\lambda_j) = \lim_{\lambda \to \lambda_j^{\pm}} \chi(x), \quad \{\{\chi\}\} = \frac{\chi^+ + \chi^-}{2}, \quad [\![\chi]\!] = \chi^+ - \chi^-,$$

the numerical flux can be defined as

$$\hat{\chi} = h_\chi(\chi^-, \chi^+), \qquad \hat{f}_\chi = \hat{f}(\chi_h^-, \chi_h^+), \qquad \hat{q} = h_q(q^-, q^+).$$

Regarding the high order derivatives we can define

$$\hat{\chi}_{r+\frac{1}{2}} = \chi^-_{r+\frac{1}{2}}, \quad \hat{q}_{r+\frac{1}{2}} = q^+_{r+\frac{1}{2}}, \quad r = 0, 1, 2, ..., N-1,$$

and the alternative choice is

$$\hat{\chi}_{r+\frac{1}{2}} = \chi^+_{r+\frac{1}{2}}, \quad \hat{q}_{r+\frac{1}{2}} = q^-_{r+\frac{1}{2}}, \quad r = 0, 1, 2, ..., N-1,$$

and we can use any monotone flux for the nonlinear part, $\hat{f}$ [32]. If we integrate by parts to (6), and, by the corresponding numerical fluxes, we replaced the fluxes at the interfaces, then we have

$$((\chi_h)_t, v)_{I_r} + (f(\chi_h)v - \sqrt{a(\chi_h)}q_h, v_x)_{I_r} + \hat{f}(\chi_h)v|_{x_r^+}^{x_{r+1}^-} - \sqrt{\hat{a}(\chi_h)}\hat{q}_h v|_{x_r^+}^{x_{r+1}^-} - \sqrt{b}(\frac{\partial s_h}{\partial x}, v)_{I_r} = 0, \tag{7}$$

$$(q_h, w(x))_{I_r} - (g(\chi_h), w_x)_{I_r} + \hat{g}(\chi_h)w|_{x_r^+}^{x_{r+1}^-} = 0, \tag{8}$$

$$(s_h, z(x))_{I_r} - \left(\Delta_{\frac{p-2}{2}} d_h, z(x)\right)_{I_r} = 0, \tag{9}$$

$$(d_h, l(x))_{I_r} - \sqrt{b}\hat{\chi}_h l|_{x_r^-}^{x_{r+1}^+} + \sqrt{b}(\chi_h, l_x)_{I_r} = 0, \tag{10}$$

$$(\chi_h(x, 0), v(x)) - (\chi_0, v(x)) = 0. \tag{11}$$

The purpose is finding $\tilde{\mathbf{w}} = \left(\tilde{\chi}, \tilde{q}, \tilde{s}, \tilde{d}\right)^T$ by exploiting the LDG method such that

$$\tilde{\chi}(x,t) = \sum_{r=1}^{N}\sum_{s=1}^{k} A_{s,r}(t)\zeta_{s,r}(x), \quad \tilde{q}(x,t) = \sum_{r=1}^{N}\sum_{s=1}^{k} B_{s,r}(t)\zeta_{s,r}(x),$$

$$\tilde{s}(x,t) = \sum_{r=1}^{N}\sum_{s=1}^{k} C_{s,r}(t)\zeta_{s,r}(x), \quad \tilde{d}(x,t) = \sum_{r=1}^{N}\sum_{s=1}^{k} D_{s,r}(t)\zeta_{s,r}(x),$$

where they are functions satisfying (7)-(10) for all $\chi, v, w, l \in \mathcal{P}^k(I_r)$, $r \in \{1, 2, ..., N\}$ and we have the initial conditions for χ, q, d and s from (11).

3. Stability

In this section, we show that LDG method is L^2-stable for nonlinear and linear equations in the semi-discrete case (rather than time discretization). To analysis the stability of LDG scheme, we need to define

$$\begin{aligned}
\Upsilon(\chi, q, s, d; v, w, z, l) = &\int_0^T \sum_{r=1}^{N} (\chi_t, v)_{I_r} \mathrm{d}t + \int_0^T \sum_{r=1}^{N-1} \left(\hat{f}v - \sqrt{\hat{a}}\hat{q}v\right)|_{x_r^+}^{x_{r+1}^-} \mathrm{d}t + \sqrt{b}\int_0^T \sum_{r=1}^{N} (s, v_x)_{I_r} \mathrm{d}t \\
&- \sqrt{b}\int_0^T \sum_{r=1}^{N-1} (\hat{s}v)|_{x_r^+}^{x_{r+1}^-} \mathrm{d}t - \int_0^T \sum_{r=1}^{N} \left(f(\chi) - \sqrt{a(\chi)}q, v_x\right)_{I_r} \mathrm{d}t \\
&+ \int_0^T \sum_{r=1}^{N} (q, w)_{I_r} \mathrm{d}t - \int_0^T \sum_{r=1}^{N} (g(\chi), w_x)_{I_r} \mathrm{d}t + \int_0^T \sum_{r=1}^{N-1} (\hat{g}(\chi)w)|_{x_r^+}^{x_{r+1}^-} \mathrm{d}t \\
&+ \int_0^T \sum_{r=1}^{N} (s, z)_{I_r} \mathrm{d}t - \int_0^T \sum_{r=1}^{N} \left(\Delta_{\frac{p-2}{2}} d, z\right)_{I_r} \mathrm{d}t + \int_0^T \sum_{r=1}^{N} (d, l)_{I_r} \mathrm{d}t
\end{aligned} \tag{12}$$

$$+\int_0^T \sum_{r=1}^{N} \sqrt{b}\,(\chi, l_x)_{I_r}\mathrm{d}t + \int_0^T \sum_{r=1}^{N-1} \sqrt{b}\hat{\chi}\, l\big|_{x_r^+}^{x_{r+1}^-}\mathrm{d}t.$$

Notice $\Upsilon(\chi, q, s, d, v, w, z, l) = 0$ for any (v, w, z, l) if (χ, q, s, d) is a solution. By considering the fluxes

$$\hat{\chi}_{r+1} = \chi_{r+1}^-, \quad \hat{q}_{r+1} = q_{r+1}^+, \quad \hat{s}_{r+1} = s_{r+1}^+, \quad \hat{g}(\chi)_{r+1} = g(\chi_{r+1}^+), \quad 1 \le r \le N-1,$$

and for the flux at the boundary, we define

$$\hat{\chi}_{N+1} = \chi(b,t), \qquad \hat{s}_{N+1} = s_{N+1}^- + \frac{\beta}{h}[\chi_{N+1}].$$

So, we yield

$$\begin{aligned}
\Upsilon(\chi, q, s, d; v, w, z, l) = &\int_0^T \sum_{r=1}^{N} (\chi_t, v)_{I_r}\mathrm{d}t - \int_0^T \sum_{r=1}^{N} (f(\chi), v_x)_{I_r}\mathrm{d}t + \sqrt{b}\int_0^T \sum_{r=1}^{N} (s, v_x)_{I_r}\mathrm{d}t \\
&+\int_0^T \sum_{r=1}^{N} \left(\sqrt{a(\chi)}q, v_x\right)_{I_r}\mathrm{d}t + \int_0^T \sum_{r=1}^{N} (q, w)_{I_r}\mathrm{d}t \\
&-\int_0^T \sum_{r=1}^{N} (g(\chi), w_x)_{I_r}\mathrm{d}t + \int_0^T \sum_{r=1}^{N} (s, z)_{I_r}\mathrm{d}t - \int_0^T \sum_{r=1}^{N} \left(\Delta_{\frac{p-2}{2}} d, z\right)_{I_r}\mathrm{d}t \\
&+\int_0^T \sum_{r=1}^{N} (d, l)_{I_r}\mathrm{d}t + \int_0^T \sum_{r=1}^{N} \sqrt{b}\,(\chi, l_x)_{I_r}\mathrm{d}t - \int_0^T \sum_{r=1}^{N-1} \hat{f}_{r+1}[v]_{r+1}\mathrm{d}t \\
&+\int_0^T \sum_{r=1}^{N-1} (\sqrt{\hat{a}}\hat{q})_{r+1}[v]_{r+1}\mathrm{d}t - \int_0^T \sum_{r=1}^{N-1} \hat{g}_{r+1}[w]_{r+1}\mathrm{d}t \\
&+\sqrt{b}\int_0^T \sum_{r=1}^{N-1} \hat{s}_{r+1}[v]_{r+1}\mathrm{d}t - \int_0^T \sum_{r=1}^{N-1} \sqrt{b}\hat{\chi}_{r+1}[l]_{r+1}\mathrm{d}t \\
&-\int_0^T \left(\hat{f}_1 v_1^+ - \hat{f}_{N+1}v_{N+1}^-\right)\mathrm{d}t + \int_0^T \left(\sqrt{\hat{a}_1}\hat{q}_1 v_1^+ - \sqrt{\hat{a}_{N+1}}\hat{q}_{N+1}v_{N+1}^-\right)\mathrm{d}t \\
&-\int_0^T \left(\hat{g}_1 w_1^+ - \hat{g}_{N+1}w_{N+1}^-\right)\mathrm{d}t + \int_0^T \left(\sqrt{b}\hat{s}_1 v_1^+ - \sqrt{b}\hat{s}_{N+1}v_{N+1}^-\right)\mathrm{d}t \\
&-\int_0^T \left(\sqrt{b}\hat{\chi}_1 l_1^+ - \hat{\chi}_{N+1}l_{N+1}^-\right)\mathrm{d}t.
\end{aligned} \tag{13}$$

Lemma 3. *By setting $(v, w, z, l) = (\chi, q, -d, s)$ in (13) and defining $F(\chi) = \int^{\chi} f(\chi)d\chi$, we achieve the following result*

$$\begin{aligned}
\Upsilon(\chi, q, s, d; \chi, q, -d, s) = &\|\chi(x,T)\|^2 - \|\chi_0\|^2 + \int_0^T \sum_{r=1}^{N} (q, q)_{I_r}\mathrm{d}t \\
&+\int_0^T \sum_{r=1}^{N} (\Delta_{\frac{p-2}{2}} d, d)_{I_r}\mathrm{d}t + \int_0^T \frac{\sqrt{b}}{h}\beta\left(\chi_{N+1}^-\right)^2\mathrm{d}t \\
&+\int_0^T \left(F(\chi)_1 - F(\chi)_{N+1} - (\hat{f}\chi)_1 + (\hat{f}\chi)_{N+1}\right)\mathrm{d}t \\
&+\int_0^T \sum_{r=1}^{N-1} \left([F(x)]_{r+1} - \hat{f}[\chi]_{r+1}\right)\mathrm{d}t.
\end{aligned}$$

Proof. If we suppose $(v,w,z,l)=(\chi,q,-d,s)$ in (13), and apply the integration by parts formula

$$I(g(\chi),q_x)_{I_r}+(g(\chi)_x,q)_{I_r}=g(\chi)q|_{x_r^+}^{x_{r+1}^-},$$
$$(s,\chi_x)_{I_r}+(s_x,\chi)_{I_r}=(s\chi)|_{x_r^+}^{x_{r+1}^-},$$

the interface condition can be obtained

$$\sum_{r=1}^{N}\left(\sqrt{a(\chi)}q,\chi_x\right)_{I_r}+\sum_{r=1}^{N}(g(\chi),q_x)_{I_r}+\sum_{r=1}^{N-1}\hat{g}(\chi)[q]_{r+1}+\sum_{r=1}^{N-1}(\sqrt{\hat{a}}\hat{q})_{r+1}[v]_{r+1}=g(\chi_1^+)q_1^+-g(\chi_{N+1}^-)q_{N+1}^-,$$

$$\sum_{r=1}^{N}\sqrt{b}(s,\chi_x)_{I_r}+\sum_{r=1}^{N}\sqrt{b}(s_x,\chi)_{I_r}+\sum_{r=1}^{N-1}\sqrt{b}s_{r+1}^+[\chi]_{r+1}+\sum_{r=1}^{N-1}\sqrt{b}\chi_{r+1}^+[s]_{r+1}=\sqrt{b}\chi_1^+s_1^+-\sqrt{b}\chi_{N+1}^-s_{N+1}^-.$$

Then we have

$$\begin{aligned}\Upsilon(\chi,q,s,d;\chi,q,-d,s)=&\int_0^T\sum_{r=1}^{N}(\chi_t,\chi)_{I_r}\mathrm{d}t-\int_0^T\sum_{r=1}^{N}(f(\chi),\chi_x)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^{N}(q,q)_{I_r}\mathrm{d}t\\&+\int_0^T\sum_{r=1}^{N}(\Delta_{\frac{p-2}{2}}d,d)_{I_r}\mathrm{d}t-\int_0^T\sum_{r=1}^{N-1}\hat{f}_{r+1}[\chi]_{r+1}\mathrm{d}t\\&+\int_0^T\frac{\sqrt{b}}{h}\varepsilon\left(\chi_{N+1}^-\right)^2\mathrm{d}t-\int_0^T\left(\hat{f}_1\hat{\chi}_1-\hat{f}_{N+1}\hat{\chi}_{N+1}\right)\mathrm{d}t.\end{aligned}\tag{14}$$

Define $F(\chi)=\displaystyle\int^{\chi}f(\chi)d\chi$, then

$$\sum_{r=1}^{N}(f(\chi),\chi_x)_{I_r}=\sum_{r=1}^{N}F(x)|_{x_r^+}^{x_{r+1}^-}=-\sum_{r=1}^{N-1}[F(x)]_{r+1}-F(\chi)_1+F(\chi)_{N+1}.\tag{15}$$

Taking into account equations (14) and (15) proves the Lemma. ☐

Theorem 4. *The semi-discrete scheme* (7)-(11) *is stable, and* $\forall T>0$ *we have* $\|u_h(x,T)\|\le\|u_0(x)\|$.

Proof. According to the properties that the monotone flux holds, $\hat{f}(\chi^-,\chi^+)$ is a non-increasing function of its second argument as well as a non-decreasing function of its first argument.

As a result, $[F(x)]_{r+1}-f[\chi]_{r+1}>0,\ 1\le r\le N-1$. By considering Galerkin orthogonality, $\Upsilon(\chi_h,q_h,s_h,d_h,\chi_h,q_h,-d_h,s_h)=0$, Lemma 3 yields

$$\begin{aligned}\|\chi(x,T)\|^2-\|\chi_0\|^2+\int_0^T\sum_{r=1}^{N}(q,q)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^{N}(\Delta_{\frac{p-2}{2}}d,d)_{I_r}\mathrm{d}t+\int_0^T\frac{\sqrt{b}}{h}\varepsilon\left(\chi_{N+1}^-\right)^2\mathrm{d}t\\+\int_0^T\left(F(\chi)_1-F(\chi)_{N+1}-(\hat{f}\chi)_1+(\hat{f}\chi)_{N+1}\right)\mathrm{d}t\le 0.\end{aligned}$$

Finally, the boundary condition leads to $\|\chi_h(x,T)\|\le\|\chi_0(x)\|$ and completes the proof. ☐

4. Error estemate for the LDG method

To estimate the error, let us assume a fractional diffusion with the Laplacian operator (when $f = 0$) also take $a \equiv 1$ and $g(\chi) = \chi$. For fractional diffusion, (7)-(11) reduce to

$$((\chi_h)_t, v)_{I_r} - (q_h, v_x)_{I_r} - \hat{q}_h v|_{x_r^+}^{x_{r+1}^-} + \sqrt{b}(p, v_x)_{I_r} - \sqrt{b}(\hat{s}v)|_{x_r^+}^{x_{r+1}^-} = 0, \tag{16}$$

$$(q_h, w(x))_{I_r} - (\chi_h, w_x)_{I_r} + \hat{\chi} w|_{x_r^+}^{x_{r+1}^-} = 0, \tag{17}$$

$$(s_h, z(x))_{I_r} - \left(\Delta_{\frac{p-2}{2}} d_h, z(x)\right)_{I_r} = 0, \tag{18}$$

$$(d_h, l(x))_{I_r} - \sqrt{b}\hat{\chi}_h l|_{x_r^-}^{x_{r+1}^+} + \sqrt{b}(\chi, l_x)_{I_r} = 0, \tag{19}$$

$$(\chi_h(x, 0), v(x)) - (\chi_0, v(x)) = 0. \tag{20}$$

Consequently, the scheme can be compactly described as

$$\begin{aligned}
\Upsilon(\chi, q, s, d; v, w, z, l) = & \int_0^T \sum_{r=1}^{N} (\chi_t, v)_{I_r} \mathrm{d}t - \int_0^T \sum_{r=1}^{N} (q, v_x)_{I_r} \mathrm{d}t + \int_0^T \sum_{r=1}^{N-1} q_{r+1}^+ [v]_{r+1} \\
& + \sqrt{b} \int_0^T \sum_{r=1}^{N} (s, v_x)_{I_r} + \int_0^T \sum_{r=1}^{N-1} \sqrt{b} s_{r+1}^+ [v]|_{x_r^+}^{x_{r+1}^-} + \int_0^T \sum_{r=1}^{N} (q, w(x))_{I_r} \\
& - \int_0^T \sum_{r=1}^{N} (\chi, w_x)_{I_r} - \int_0^T \sum_{r=1}^{N-1} \hat{\chi}_{r+1}^+ [w]_{r+1} + \int_0^T \sum_{r=1}^{N} (s, z(x))_{I_r} \\
& - \int_0^T \sum_{r=1}^{N} (\Delta_{\frac{p-2}{2}} d, z(x)) + \int_0^T \sum_{r=1}^{N} (d, l(x))_{I_r} \\
& + \int_0^T \sum_{r=0}^{N} \sqrt{b} (\chi, l_x)_{I_r} + \sqrt{b} \int_0^T \sum_{r=1}^{N-1} \hat{\chi}_{r+1}^+ [l]_{r+1}|_{x_r^-}^{x_{r+1}^+} \\
& + \sqrt{b} \int_0^T s_1^+ v_1^+ \mathrm{d}t + \frac{\sqrt{b}\beta}{h} \int_0^T \chi_{N+1}^- v_{N+1}^- \mathrm{d}t - \sqrt{b} \int_0^T s_{N+1}^- v_{N+1}^- \mathrm{d}t.
\end{aligned} \tag{21}$$

We define special projection $\mathbb{S}^{\pm}$ into V^k which holds

$$\int_{I_r} \left(\mathbb{S}^{\pm} e(x) - e(x)\right) \zeta_{jr}(x) \mathrm{d}x = 0. \tag{22}$$

Where $r = 1, 2, .., N, \;\; j = \{0, 1, 2, ..., k-1\}$, and $\mathbb{S}^{\pm}\chi_{r+1} = \chi(x_{r+1}^{\pm})$. Denote $e_\chi = \chi - \chi_h, \;\; e_s = s - s_h, \;\; e_q = q - q_h$, and $e_d = d - d_h$, then $\mathbb{S}^- e_\chi = \mathbb{S}^- \chi - \chi_h, \;\; \mathbb{S}^+ e_s = \mathbb{S}^+ s - s_h, \;\; \mathbb{S}^+ e_q = \mathbb{S}^+ q - q_h$, and $\mathbb{P} e_d = \mathbb{P} d - d_h$ for all $(v, w, z, d) \in H^1(\Omega, \mathcal{T}) \times L^2(\Omega, \mathcal{T}) \times L^2(\Omega, \mathcal{T}) \times L^2(\Omega, \mathcal{T})$,

$$\Upsilon(\chi, q, s, d; v, w, z, l) = \mathbb{S}(v, w, z, l). \tag{23}$$

Hence, $\Upsilon\left(e_\chi, e_q, e_s, e_d; v, w, z, l\right) = 0$ and we gain

$$\begin{aligned}
\Upsilon & \left(\mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, \mathbb{S}^+ e_s, \mathbb{P} e_d; \mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, -\mathbb{P} e_d, \mathbb{S}^+ e_s\right) \\
& = \Upsilon\left(\mathbb{S}^- e_\chi - e_\chi, \mathbb{S}^+ e_q - e_q, \mathbb{S}^+ e_s - e_s, \mathbb{P} e_d - e_d; \mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, -\mathbb{P} e_d, \mathbb{S}^+ e_s\right) \\
& = \Upsilon\left(\mathbb{S}^- \chi - \chi, \mathbb{S}^+ q - \mathrm{q}, \mathbb{S}^+ s - s, \mathbb{P} d - d; \mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, -\mathbb{P} e_d, \mathbb{S}^+ e_s\right).
\end{aligned}$$

Substitute $(\mathbb{S}^- \chi - \chi, \mathbb{S}^+ q - \mathrm{q}, \mathbb{S}^+ s - s, \mathbb{P} d - d; \mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, -\mathbb{P} e_d, \mathbb{S}^+ e_s)$ into (21) to achieve the following Lemma.

Lemma 5. *For the compact from, (21), we achieve*

$$\Upsilon\left(\mathbb{S}^-\chi-\chi,\mathbb{S}^+q-\mathrm{q},\mathbb{S}^+s-s,\mathbb{P}d-d;\mathbb{S}^-e_\chi,\mathbb{S}^+e_q,-\mathbb{P}e_d,\mathbb{S}^+e_s\right)$$
$$\le\int_0^T\sum_{r=1}^N\left((\mathbb{S}^-\chi)_t-\chi_t,\mathbb{S}^-e_\chi\right)_{I_r}\mathrm{d}t+C_{T,a,b}h^{2k+2}+\frac{1}{C_{T,a,b}}\int_0^T\sum_{r=1}^N\|\mathbb{P}e_q\|_{I_r}^2\,\mathrm{d}t$$
$$+\int_0^T\frac{\sqrt{b}\beta}{h}|(\mathbb{S}^-e_\chi)_{N+1}|^2\mathrm{d}t+\int_0^T\sum_{r=1}^N\left\|\mathbb{S}^+e_q\right\|_{I_r}^2\mathrm{d}t,$$

where $C_{T,a,b}$ is independent of h, but may depend on T and Ω.

Proof. From (21) we have

$$\Upsilon\left(\mathbb{S}^-\chi-\chi,\mathbb{S}^+q-\mathrm{q},\mathbb{S}^+s-s,\mathbb{P}d-d;\mathbb{S}^-e_\chi,\mathbb{S}^+e_q,-\mathbb{P}e_d,\mathbb{S}^+e_s\right)$$
$$=\int_0^T\sum_{r=1}^N\left((\mathbb{S}^-\chi)_t-\chi_t,\mathbb{S}^-e_\chi\right)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^N\left((\mathbb{S}^+q-q,(\mathbb{S}^-e_\chi)_x\right)_{I_r}\mathrm{d}t+\sqrt{b}\int_0^T\sum_{r=0}^N\left((\mathbb{S}^+s-s,(\mathbb{S}^-e_\chi)_x\right)_{I_r}\mathrm{d}t$$
$$+\int_0^T\sum_{r=1}^N\left((\mathbb{S}^+q-q,\mathbb{S}^+e_q\right)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^N\left((\mathbb{S}^-\chi-\chi,(\mathbb{S}^+e_q)_x\right)_{I_r}\mathrm{d}t-\int_0^T\sum_{r=1}^N\left(\mathbb{S}^+s-s,\mathbb{P}e_d\right)_{I_r}\mathrm{d}t$$
$$+\int_0^T\sum_{r=1}^N\left(\Delta_{\frac{p-2}{2}}(\mathbb{P}d-d),\mathbb{P}e_d\right)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^N\left(\mathbb{P}d-d,\mathbb{S}^+e_s\right)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^N\left(\mathbb{S}^-\chi-\chi,(\mathbb{S}^+e_s)_x\right)_{I_r}\mathrm{d}t$$
$$-\int_0^T\sum_{r=1}^N(\mathbb{S}^+q-q)_{r+1}^+[\mathbb{S}^-e_\chi]_{r+1}\mathrm{d}t-\sqrt{b}\int_0^T\sum_{r=1}^{N-1}(\mathbb{S}^+s-s)_{r+1}^+[\mathbb{S}^-e_\chi]_{r+1}\mathrm{d}t$$
$$-\int_0^T\sum_{r=1}^N(\mathbb{S}^-\chi-\chi)_{r+1}^-[\mathbb{S}^+e_q]_{r+1}\mathrm{d}t-\sqrt{b}\int_0^T\sum_{r=1}^{N-1}(\mathbb{S}^-\chi-\chi)_{r+1}^-[\mathbb{S}^+e_s]_{r+1}\mathrm{d}t$$
$$+\sqrt{b}\int_0^T(\mathbb{S}^+s-s)_1^+[\mathbb{S}^-e_\chi^+]_1\mathrm{d}t+\frac{\sqrt{b}\beta}{h}\int_0^T(\mathbb{S}^-\chi-\chi)_{N+1}^+[\mathbb{S}^-e_\chi^-]_{N+1}\mathrm{d}t-\sqrt{b}\int_0^T(\mathbb{S}^+s-s)_{N+1}^-[\mathbb{S}^-e_\chi^-]_{N+1}\mathrm{d}t.$$

Since $(\mathbb{S}^+e_s)_x\in\mathcal{P}^{k-1}$, $(\mathbb{S}^-e_\chi)_x\in\mathcal{P}^{k-1}$, $(\mathbb{S}^+e_q)_x\in\mathcal{P}^{k-1}$, $\mathbb{P}e_d\in\mathcal{P}^k$, by considering the properties of the projection $\mathbb{S}^\pm$ and $\mathbb{P}$: $\left(\mathbb{S}^+q-q,(\mathbb{S}^-e_\chi)_x\right)_{I_r}=0$, $(\mathbb{S}^+s-s,(\mathbb{S}^-e_\chi)_x)_{I_r}=0$, $((\mathbb{S}^-\chi-\chi,(\mathbb{S}^+e_s)_x)_{I_r}=0$, $(\mathbb{P}d-d,\mathbb{S}^+e_s)_{I_r}=0$, $((\mathbb{P}d-d,(\mathbb{S}^+e_s)_x)_{I_r}=0,(\mathbb{S}^+s-s)_{r+1}=0$, and $(\mathbb{S}^-\chi-\chi)_{r+1}=0$, therefore

$$\Upsilon\left(\mathbb{S}^-\chi-\chi,\mathbb{S}^+q-\mathrm{q},\mathbb{S}^+s-s,\mathbb{P}d-d;\mathbb{S}^-e_\chi,\mathbb{S}^+e_q,-\mathbb{P}e_d,\mathbb{S}^+e_s\right)$$
$$=\int_0^T\sum_{r=1}^N\left((\mathbb{S}^-\chi)_t-\chi_t,\mathbb{S}^-e_\chi\right)_{I_r}\mathrm{d}t+\int_0^T\sum_{r=1}^N\left(\mathbb{S}^+q-q,\mathbb{S}^+e_q\right)_{I_r}\mathrm{d}t$$
$$+\int_0^T\sum_{r=1}^N\left(\Delta_{\frac{p-2}{2}}(\mathbb{P}d-d)-(\mathbb{S}^+s-s),\mathbb{P}e_d\right)_{I_r}\mathrm{d}t-\sqrt{b}\int_0^T(\mathbb{S}^+s-s^-)_{N+1}[\mathbb{S}^-e_\chi]_{N+1}^-\mathrm{d}t.$$

From Lemma 2 and the projection property we can conclude

$$\left\|\Delta_{p/2}(\mathbb{P}d-d)-(\mathbb{S}^+s-s)\right\|\le Ch^{k+1}.$$

Combining this with Young's inequality and we obtain

$$\Upsilon\left(\mathbb{S}^-\chi-\chi,\mathbb{S}^+q-\mathrm{q},\mathbb{S}^+s-s,\mathbb{P}d-d;\mathbb{S}^-e_\chi,\mathbb{S}^+e_q,-\mathbb{P}e_d,\mathbb{S}^+e_s\right)$$

$$\leq \int_0^T \sum_{r=1}^{N} \left((\mathbb{S}^- \chi)_t - u_t, \mathbb{S}^- e_\chi \right)_{I_r} \mathrm{d}t + C_{T,a,b} h^{2k+2} + \frac{1}{C_{T,a,b}} \int_0^T \sum_{r=1}^{N} \|\mathbb{P} e_q\|_{I_r}^2 \,\mathrm{d}t$$

$$+ \int_0^T \frac{\sqrt{b}\beta}{h} |(\mathbb{S}^- e_\chi)_{N+1}|^2 \mathrm{d}t + \int_0^T \sum_{r=1}^{N} \left\|\mathbb{S}^+ e_q\right\|_{I_r}^2 \mathrm{d}t,$$

that it proves the Lemma. □

Theorem 6. *Suppose χ be a exact solution to* (2) *which is sufficiently smooth in $\Omega \subset \mathbb{R}$ such that $f(\chi) = 0$. For small enough h, assuming χ_h be the numerical solution of the semi-discrete LDG scheme* (7)-(11)*, the following error can be estimated*

$$\|\chi - \chi_h\| \leq C h^{k+1},$$

where C is a constant independent of h.

Proof. Recalling Lemma 3 and from the initial error $\|S^- e_\chi(0)\| = 0$ we have

$$\Upsilon \left(\mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, \mathbb{S}^+ e_s, \mathbb{P} e_d; \mathbb{S}^- e_\chi, \mathbb{S}^+ e_q, -\mathbb{P} e_d, \mathbb{S}^+ e_s \right)$$

$$= \frac{1}{2} \left\|\mathbb{S}^- e_\chi(T)\right\|^2 + \int_0^T \sum_{r=0}^{N} \left(\Delta_{\frac{p-2}{2}}(\mathbb{P} e_d), \mathbb{P} e_d \right)_{I_r} \mathrm{d}t$$

$$+ \int_0^T \sum_{r=1}^{N} \left\|\mathbb{S}^+ e_q\right\|_{I_r}^2 \mathrm{d}t + \int_0^T \frac{\sqrt{b}\beta}{h} |(\mathbb{S}^- e_\chi)_{N+1}|^2 \mathrm{d}t.$$

Recalling Lemma 5, we have

$$\frac{1}{2} \left\|\mathbb{S}^- e_\chi(T)\right\|^2 + \int_0^T \sum_{r=0}^{N} \left(\Delta_{\frac{p-2}{2}}(\mathbb{P} e_d), \mathbb{P} e_d \right)_{I_r} \mathrm{d}t.$$

$$\leq \int_0^T \sum_{r=0}^{N} \left((\mathbb{S}^- \chi)_t - \chi_t, \mathbb{S}^- e_\chi \right)_{I_r} \mathrm{d}t + C_{T,a,b} h^{2k+2} + \frac{1}{C_{T,a,b}} \int_0^T \sum_{r=0}^{N} \|\mathbb{P} e_q\|_{I_r}^2 \mathrm{d}t.$$

Combining this with the fractional Poincaré-Friedrichs Lemma in [36], we get

$$\frac{1}{2} \left\|\mathbb{S}^- e_\chi(T)\right\|^2 \leq \int_0^T \sum_{r=0}^{N} \left((\mathbb{S}^- \chi)_t - \chi_t, \mathbb{S}^- e_\chi \right)_{I_r} \mathrm{d}t + C_{T,a,b} h^{2k+2}$$

The error related to the projection error, proves the theorem. □

5. Numerical results

Three example problems have been examined by MATLAB programming to illustrate the efficiency and accuracy of the method for solving the general problem with nonlinear diffusion term and nonlinear Burgers' equations. The accuracy and efficiency of the proposed method is shown by employing L^2-error, $Err_h = \|\chi - \chi_h\|_2$, the approximate rate of convergence, $O(N,k) = \dfrac{log(\mathrm{Err}_h) - log(\mathrm{Err}_{h/m})}{log(m)}$, and maximum relative error, $MRE = \max\limits_{1 \leqslant i \leqslant N} \left| \dfrac{\chi^i - \chi_h^i}{\chi^i} \right|$, where χ^i and χ_h^i denote the ith components of the exact and approximated solutions, respectively.

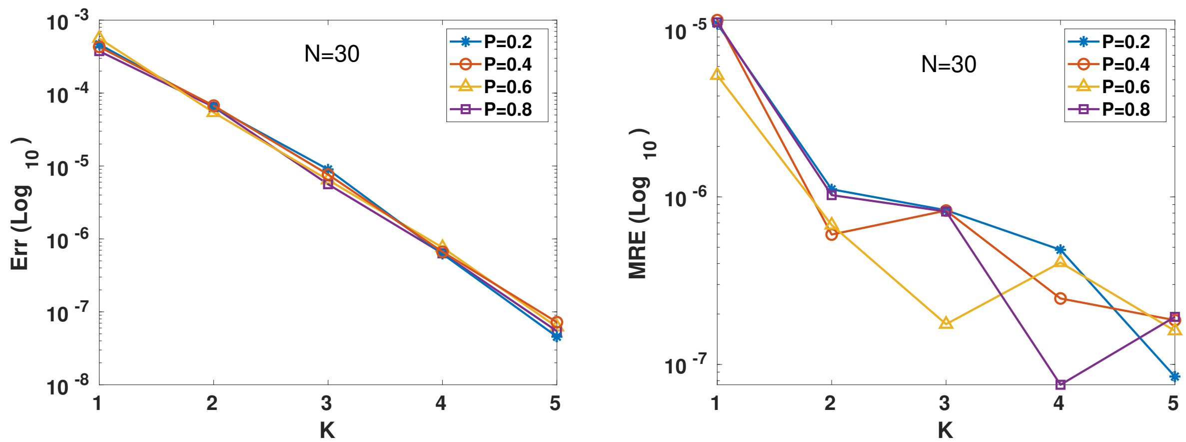

Figure 1: Norm errors (left) and maximum relative errors (right) for the numerical solutions of Eq. (2), Example 5.1, with time $T = 1$ and various p versus polynomial order K with $N = 30$.

Example 5.1. Consider the following parameters from Equation (2)

$$f = -2^{1-p}\left(x^2\left(3p^2 - 12p + 9\right) + x\left(4p^2 - 12p + 8\right)\right),$$
$$a = \frac{x^3}{3} + 2^{-p}\left(3p^2 - 15p + 18\right)e^{-t}x^2,$$
$$b = p^3 - 6p^2 + 11p - 6.$$

The exact solution for $p \in (0, 1)$ and $x \in (0, 2)$ is $\chi = e^t x^3$.

We solved this problem for $T = 1$ by using the LDG method. The results in Figs. 1 shows that the errors of the approximate solution decay exponentially. Thus, we can conclude that method that the LDG method guarantees an exponential convergence speed. However, maximum relative errors do not illustrate unique patterns, for instance, sometimes the results are better for less k (Fig. 1, right). Moreover, Table 1 verifies the convergence rate h^{k+1}.

Table 1: The norm errors, Err, maximum relative errors, MRE, and the convergence rates, $O(N, k)$, of LDG method for different k and N for Example 5.1 when $T = 1$ and $p = 0.4$.

		$k = 2$			$k = 4$		
p	N	Err_h	$O(N,k)$	MRE	Err_h	$O(N,k)$	MRE
0.4	10	5.35e-03	-	6.28e-05	2.88e-07	-	3.57e-09
	20	8.99e-04	2.29	8.03e-06	1.78e-08	4.01	2.63e-10
	40	1.05e-08	3.09	9.64e-09	1.37e-09	3.70	9.73e-10

Example 5.2. Consider the following fractional diffusion equation with the fractional Laplacian [36],

$$\frac{\partial \chi(x,t)}{\partial t} = -(-\Delta)^{\frac{p}{2}}\chi(x,t) + \rho(x,t), \; (x,t) \in [0,1] \times (0, 0.5],$$
$$\chi(x, 0) = \chi_0(x),$$

with the initial conditions

$$\chi_0(x) = \begin{cases} x^6(1-x^6), & 0 \le x \le 1, \\ 0, & otherwise, \end{cases}$$

and the source term (or the external force) $\rho(x,t) = e^{-t}(-\chi_0(x) + (-\Delta)^{\frac{p}{2}}\chi_0(x))$.
The exact solution is $\chi(x,t) = e^{-t}x^6(1-x^6)$. We solve this problem for different values of p at a fixed time $t = 1$. Table. 2 shows the error norms for different conditions and confirms the convergence order h^{k+1}.

Table 2: The error, Err, and convergence order, $O(N,k)$, of LDG method for example 5.2.

		$k=1$		$k=2$		$k=3$	
p	N	Err_h	$O(N,k)$	Err_h	$O(N,k)$	Err_h	$O(N,k)$
1.3	10	1.27e-04	-	2.34e-05	-	3.22e-06	-
	20	3.22e-05	1.97	2.94e-06	2.99	2.08e-07	3.96
	40	8.14e-06	1.98	3.61e-07	3.02	1.33e-8	3.97
	80	1.98e-06	2.03	4.45e-08	3.02	8.20e-10	4.01
1.5	10	1.71e-05	-	3.21e-06	-	4.22e-07	-
	20	4.18e-06	2.03	3.92e-04	3.03	2.71e-08	3.96
	40	1.01e-06	2.02	4.86e-08	3.01	1.68e-09	4.01
	80	2.52e-07	2.00	5.97e-09	3.02	1.04e-10	4.01
1.7	10	2.37e-06	-	4.67e-07	-	3.22e-08	-
	20	5.88e-07	2.01	4.67e-07	3.00	3.22e-08	3.99
	40	1.48e-07	1.98	7.21e-09	3.01	1.25e-10	4.01
	80	3.68e-08	2.00	9.04e-10	2.99	7.70e-12	4.02

Example 5.3. Consider the nonlinear Burgers' equation as follow [36]

$$\begin{cases} \frac{\partial\chi(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\chi^2(x,t)}{2}\right) = a\left(-(-\Delta)^{\frac{p}{2}}\right)\chi(x,t) + Z(x,t), & (x,t) \in [-2,2] \times (0, 0.5], \\ \chi(x,0) = \chi_0(x), \quad x \in [-2,2], \end{cases}$$

with the discontinuous initial condition

$$\chi_0(x) = \begin{cases} \frac{(1-x^2)^4}{10}, & -1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

If we set the parameter $a = 1$ and consider the source term as

$$Z(x,t) = e^{-t}(-\chi_0(x) + e^{-t}(\chi_0(x)\chi_0'(x) + (-\Delta)^{\frac{p}{2}}\chi_0(x)),$$

we can obtain the exact solution as follow

$$\chi(x,t) = \begin{cases} \frac{e^{-t}(1-x^2)^4}{10}, & -1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

We solve the equation for different orders of the operator, p, polynomial orders, k, and numbers of elements, N. The errors and rate of convergence in Table 3 show the convergence rate, $O(h^{k+1})$, for $1 < p < 2$. In this table, the highlighted columns recall the results given from [36] to compare with the proposed method in this paper. Thus, we can deduce that the LGD methods with Spline polynomials improve accuracy (by increasing computation costs) than GD methods with Legendre polynomials.

Table 3: Comparison of the norm error and the convergence rate of LDG method and DG method [36] versus k, N, and p for Example 5.3.

		$k=1$				$k=2$				$k=3$			
		LDG method		DG method		LDG method		DG method		LDG method		DG method	
p	N	Err_h	$O(N,k)$	Err_h	$O(N,k)$	Err_h	$O(N,k)$	Err_h	$O(N,k)$	Err_h	$O(N,k)$	Err_h	$O(N,k)$
1.1	10	1.65e-07	-	1.10e-03	-	3.82e-09	-	6.53e-05	-	3.45e-10	-	5.94e-06	-
	20	3.98e-08	2.05	1.00e-05	1.97	4.82e-10	2.99	1.00e-05	2.70	2.22e-11	4.00	4.00e-07	3.89
	40	9.84e-09	2.02	6.90e-05	2.03	5.93e-11	3.02	1.33e-06	2.94	1.38e-12	4.00	2.58e-08	3.95
1.5	10	1.65e-08	-	8.89e-04	-	6.45e-09	-	6.71e-05	-	5.45e-10	-	4.91e-06	-
	20	4.08e-09	1.98	2.15e-04	2.05	8.10e-10	2.99	8.62e-06	2.96	3.45e-11	3.98	3.34e-07	3.88
	40	9.99e-10	2.02	5.28e-03	2.02	9.98e-11	3.01	1.09e-06	2.99	2.15e-12	4.00	2.16e-08	3.99
1.8	10	1.28e-08	-	8.43e-04	-	3.28e-09	-	6.78e-05	-	5.45e-10	-	4.80e-06	-
	20	3.22e-09	1.99	2.09e-04	2.01	4.12e-10	2.99	8.59e-06	2.98	3.36e-11	4.01	3.32e-07	3.85
	40	8.02e-10	2.01	5.20e-05	2.00	5.08e-11	3.02	1.08e-06	2.99	2.11e-12	3.99	2.20e-08	3.94

Example 5.4. Consider the nonlinear Burgers equation as follow

$$\frac{\partial \chi(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\chi^2(x,t)}{2}\right) = \left(-(-\Delta)^{\frac{p}{2}}\right)\chi(x,t), \quad \text{in } [0,2]\times(0,1], \tag{24}$$

with the discontinuous initial condition

$$\chi_0(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

We solve the equation (24) for several different values of p at a fixed time $t = 1$. Fig. 2 illustrates the error norms for different conditions. The numerical results show the exponential convergence speed concerning polynomial order k (when $N = 20$).

Conclusions

In this paper, we have investigated the LDG method on a non-linear Convection-Diffusion problem. A fractional Laplacian represents the fractional diffusion of the problem. Spline interpolations have been considered to obtain a high convergence order. Subsequently, the problem has been converted to a system with a lower order by replacing a combination of first order derivatives and integrals as the fractional Laplacian. The equation problem has been considered in the domain Ω satisfying homogeneous boundary conditions. The stability of the proposed LDG

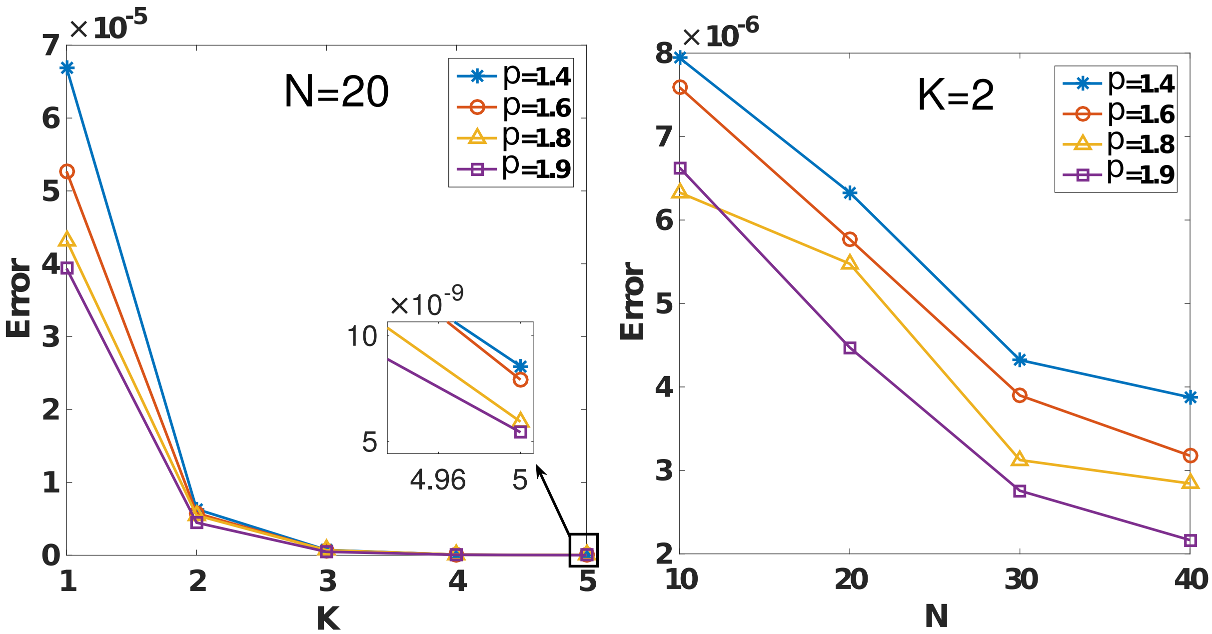

Figure 2: A comparison of the estimated errors (Err) for the numerical solutions of Eq. (24), Example 5.4, with time $T = 1$ and various p versus polynomial order k with $N = 20$ (left) and the number of elements N with $k = 2$ (right).

method has been derived and its error has been estimated of the convergence order h^{k+1}. By providing numerical examples, the accuracy of the analytical results has been illustrated.

The studied model with the nonlocal operator can have broad applications in modeling, for example, molecular biology, dislocation dynamics, and finance. The advantage of the LDG algorithm is the solving generalized form of the fractional convection-diffusion equation with the nonlinear diffusion term and fractional Laplacian order $p \in (0, 2)$. Besides, using Spline polynomials aims to elevate the accuracy of the method and achieves convergence order h^{k+1} with general monotone flux for the nonlinear term. However, the proposed procedure for reducing the original problem PDE to a simpler system with the first order leads to more computation costs. Thus, there is still a great demand for finding the optimal numerical method of solving complex nonlinear equations with nonlocal operators.

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Conflict of Interest

The authors declare no conflict of interest.

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