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A simple undecidable problem for free groups ☆

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ABSTRACT

Let F_n denote the free group on n generators. It is shown to be undecidable for two morphisms $g, h\colon F_n \to F_2$ and a generator element a of F_2, whether or not there exists an element $w \in F_n$ such that $g(w) = a$ and $h(w) = 1$, where 1 is the identity element.

As is well known the Post Correspondence Problem is undecidable for free monoids, i.e., it is undecidable for pairs of morphisms $g, h\colon A^* \to B^*$ whether or not there exists a nonempty word w such that $g(w) = h(w)$; see Post [5]. In other words, it is undecidable if the *equalizer*

$$Eq(g,h) = \{w \mid g(w) = h(w)\}$$

is nontrivial.

A more restricted version of the problem, where the images of the morphisms are fixed, is trivial for free monoids, i.e., one can easily check for two morphisms g, h and fixed words x, y whether or not there exists a word w such that $g(w) = x$ and $h(w) = y$. Indeed, the sets $g^{-1}(x)$ and $h^{-1}(y)$ are regular, and so is their intersection. Moreover, the emptiness problem is decidable for regular languages.

For free groups this problem is more complicated, due to cancellations by the inverse elements. Indeed, we shall show that the restricted problem becomes undecidable for very simple elements $x = a$ and $y = 1$, where a is a generator of the free group and 1 is the identity element.

Denote by F_n the free group of the reduced words freely generated by the n elements $A_n = \{a_1, a_2, \dots, a_n\}$. For a subset $S \subseteq F_n$, let $\langle S \rangle$ be the subgroup of F_n generated by S. Clearly, $\langle S \rangle$ is a free subgroup of F_n. For this and other properties of free groups, we refer to the book by Lyndon and Schupp [2].

The following theorem is due to Miller [4] for F_6, and Schupp [6] for F_2. The result is based on the work of Mikhaĭlova [3].

Theorem 1. *It is undecidable for finite subsets $S \subseteq F_2 \times F_2$ whether or not $\langle S \rangle = F_2 \times F_2$.*

Now, $\langle S \rangle = F_2 \times F_2$ if and only if $(1, a_i)$ and $(a_i, 1)$ are in $\langle S \rangle$ for both $i = 1, 2$. Since there are only finitely many of these pairs, we have, by symmetry, the following corollary.

Corollary 1. *It is undecidable for finite subsets $S \subseteq F_2 \times F_2$ and generators a of F_2 whether or not $(a, 1) \in \langle S \rangle$.*

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Each finite subset $S \subseteq F_2 \times F_2$ can be identified by a pair of morphisms $g, h : F_n \to F_2$, where n is the cardinality of S, say $S = \{(u_1, v_1), \dots, (u_n, v_n)\}$, by letting

$g(a_i) = u_i$ and $h(a_i) = v_i$ for the generators $a_i \in A_n$.

Corollary 2. *It is undecidable for pairs of morphisms* $g, h : F_n \to F_2$ *whether or not* $\langle (g(a), h(a)) \mid a \in A_n \rangle = F_2 \times F_2$, *i.e., whether or not*

$$\{(g(w), h(w)) \mid w \in F_n\} = F_2 \times F_2.$$

Corollary 3. *It is undecidable for pairs of morphisms* $g, h : F_n \to F_2$ *and a generator* a *of* F_2 *whether or not there exists an element* $w \in F_n$ *such that* $g(w) = a$ *and* $h(w) = 1$.

The *equalizer* of two morphisms $g, h : F_n \to F_m$ is defined by

$$Eq(g, h) = \{w \mid g(w) = h(w)\}.$$

It is a free subgroup of F_n.

One of the main open questions for the equalizers of free groups was posed by Stallings [7]: if both $g, h : F_n \to F_m$ are injective, does the generator set of $Eq(g, h)$ have at most n elements?

See Logan [1] for rank two free groups.

The original statement by Post [5] for word monoids was that there exists finitely generated free monoids for which the Correspondence Problem undecidable. In a more general setting, we can consider arbitrary (finitely generated) monoids M. Then the *Post Correspondence Problem* for M, asks whether, for given pairs (g, h) of morphisms $g, h : M \to M$, there exists an element $x \in M$ such that $g(x) = h(x)$ with $x \neq 1$, the identity element of M.

Problem. Characterize those monoids for which the Post Correspondence Problem is decidable.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Alan D. Logan, The equalizer conjecture for the free group of rank two, Q. J. Math. 73 (2) (2021) 777–793.
[2] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 89, 1977.
[3] K.A. Mikhaĭlova, The occurrence problem for free products of groups, Dokl. Akad. Nauk SSSR 127 (1959) 746–748.
[4] Charles F. Miller III, On Group-Theoretic Decision Problems and Their Classification, Ann. Math. Stud., vol. 68, Princeton University Press, Princeton, NJ, 1971.
[5] Emil Leon Post, A variant of a recursively unsolvable problem, Bull. Am. Math. Soc. 52 (1946) 264–268.
[6] Paul E. Schupp, Embeddings into simple groups, J. Lond. Math. Soc., II. Ser. 13 (1976) 90–94.
[7] J.R. Stallings, Finiteness properties of matrix representations, Ann. Math. 124 (1986) 337–346.