



# Regularity theory for non-autonomous problems with a priori assumptions

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Received: 23 May 2023 / Accepted: 20 September 2023 / Published online: 21 October 2023  
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## Abstract

We study weak solutions and minimizers  $u$  of the non-autonomous problems  $\operatorname{div} A(x, Du) = 0$  and  $\min_v \int_{\Omega} F(x, Dv) dx$  with quasi-isotropic  $(p, q)$ -growth. We consider the case that  $u$  is bounded, Hölder continuous or lies in a Lebesgue space and establish a sharp connection between assumptions on  $A$  or  $F$  and the corresponding norm of  $u$ . We prove a Sobolev–Poincaré inequality, higher integrability and the Hölder continuity of  $u$  and  $Du$ . Our proofs are optimized and streamlined versions of earlier research that can more readily be further extended to other settings. Connections between assumptions on  $A$  or  $F$  and assumptions on  $u$  are known for the double phase energy  $F(x, \xi) = |\xi|^p + a(x)|\xi|^q$ . We obtain slightly better results even in this special case. Furthermore, we also cover perturbed variable exponent, Orlicz variable exponent, degenerate double phase, Orlicz double phase, triple phase, double variable exponent as well as variable exponent double phase energies and the results are new in most of these special cases.

**Mathematics Subject Classification** 35B65 · 35A15 · 35J62 · 46E35 · 49N60

## 1 Introduction

We consider the divergence form, quasilinear elliptic equation

$$\operatorname{div} A(x, Du) = 0 \quad \text{in } \Omega, \quad (\operatorname{div} A)$$

and the corresponding  $F$ -energy minimization

$$\min_u \int_{\Omega} F(x, Du) dx, \quad (\min F)$$

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Communicated by Laszlo Szekelyhidi.

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where  $A$  and  $F$  have quasi-isotropic  $(p, q)$ -growth (see Definition 4.1). Since we allow  $A$  and  $F$  to depend on  $x$ , these are *non-autonomous* problems. The strategy for dealing with non-autonomous problems is often the reduction to and approximation with autonomous problems, such as the  $p$ -power energy  $F(\xi) := |\xi|^p$ ,  $p \in (1, \infty)$ , and the  $p$ -Laplace equation with  $A(\xi) := |\xi|^{p-2}\xi$ . The maximal regularity of weak solutions already to the  $p$ -Laplace equation when  $p \neq 2$  is  $C^{1,\alpha}$  for some  $\alpha \in (0, 1)$  (e.g., [23, 29, 43, 46, 52]) and this is the objective also in more general cases, including in this article. The approximation technique is often used to deal with Marcellini’s [47]  $(p, q)$ -growth energies,  $|\xi|^p \lesssim F(\xi) \lesssim |\xi|^q + 1$  and  $1 < p \leq q$ , provided that  $\frac{q}{p}$  is close to 1, see, e.g., [9, 10, 21, 22, 48].

To explain the objective of the current paper we consider the *double phase functional*  $F(x, \xi) := |\xi|^p + a(x)|\xi|^q$  with  $1 < p \leq q$  and  $a : \Omega \rightarrow [0, L_\omega]$ , which is a special case of  $(p, q)$ -growth. This model was first studied by Zhikov [53, 54] in the 1980’s and has recently enjoyed a resurgence after a series of papers by Baroni, Colombo and Mingione [6–8, 15–17]. They studied the relationship between the parameters  $p$  and  $q$  and the Hölder-exponent  $\alpha$  of  $a$  and established maximal regularity of the minimizer  $u$  in the following three cases of a priori information:

- (ap1)  $u \in W^{1,p}(\Omega)$  and  $q - p \leq \frac{p}{n}\alpha$
- (ap2)  $u \in L^\infty(\Omega)$  and  $q - p \leq \alpha$
- (ap3)  $u \in C^{0,\gamma}(\Omega)$  and  $q - p < \frac{1}{1-\gamma}\alpha$

Furthermore, in the first two cases the inequality is sharp in the sense that there exist counterexamples to regularity which fail the inequality arbitrarily little [5, 28]. The case of equality in (ap3) is an open problem. Ok [50] added to these a fourth, likewise sharp, case:

- (ap4)  $u \in L^{s^*}(\Omega)$  and  $q - p \leq \frac{s}{n}\alpha$

In view of the Sobolev embedding when  $s < n$ ,  $s = n$  and  $s > n$ , this suggests the unifying, albeit slightly stronger, assumption  $u \in W^{1,s}(\Omega)$  and  $q - p \leq \frac{s}{n}\alpha$ .

While the relationship between a priori information on  $u$  and the conditions for double phase  $F$  are quite well understood, this is not the case for the wide range of recently introduced double phase variants, which extend it or combine it with the variable exponent case  $F(x, \xi) = |\xi|^{p(x)}$  [24, 51]. These variants include perturbed variable exponent, Orlicz variable exponent, degenerate double phase, Orlicz double phase, triple phase, double variable exponent and variable exponent double phase. See Corollary 1.1 for the corresponding expressions  $F$  and Table 1 for examples of our assumptions in some of these cases. We refer to [41] for references up to 2020 and [3, 4, 18, 20, 30, 32, 45, 49] for some more recent advances on variants of the variable exponent and double phase models.

In most of the special cases, both lower and maximal regularity remain unstudied under assumptions (ap2)–(ap4). Recently, Baasandorj and Byun [2] proved maximal regularity of the Orlicz triple phase case in a massive paper. Rather than study each case individually, we introduced an approach based on generalized Orlicz spaces in [41] and proved maximal regularity for minimizers when  $F(x, \xi) = F(x, |\xi|)$  has so-called *Uhlenbeck structure*. In [42] we extended the results to weak solutions and minimizers of problems with  $(p, q)$ -growth without the Uhlenbeck restriction. In both articles we only considered the assumption  $u \in W^{1,\varphi}(\Omega)$  corresponding to case (ap1). In this article we cover all the different assumptions from cases (ap1)–(ap4), including as special cases all the double phase variants listed in the previous paragraph.

We build on the harmonic approximation approach from [8]. Our method is more streamlined and we are even able to improve the results in the double phase case slightly by introducing the following version of (ap2), which is natural to expect based on the intuition

of the Sobolev embedding, and a version of (ap3) with equality provided we have vanishing Hölder continuity:

$$\begin{aligned} \text{(ap2')} \quad & u \in BMO(\Omega) \text{ and } q - p \leq \alpha \\ \text{(ap3')} \quad & u \in VC^{0,\gamma}(\Omega) \text{ and } q - p \leq \frac{1}{1-\gamma}\alpha \end{aligned}$$

This article represents a substantial generalization and unification of prior theory. We expect that our optimized methods can more readily be further extended to other settings.

In recent years, many papers consider bounded weak solutions or minimizers (i.e. Case (ap2)), see for instance [2, 12, 16, 34]. The boundedness can be naturally obtained from the maximum principle for bounded Dirichlet boundary value problems, and is thus a fundamental assumption. The following special case of Corollary 5.14 showcases our results for  $L^\infty$ . We emphasize that even many of these special case results are new and that our main results, Theorems 5.3 and 5.11 and Corollary 5.14, also cover other a priori assumptions and structures.

**Corollary 1.1** (Bounded minimizers in special cases) *Let  $1 < p < \min\{q, r\}$ , the variable exponents  $p(x)$  and  $q(x)$  with  $p(x) \leq q(x)$  be Hölder continuous and bounded away from 1 and  $\infty$ , and  $a \in C^{0,\alpha_a}$  and  $b \in C^{0,\alpha_b}$  be non-negative and bounded. Assume that  $F(x, \xi) = f(x, |\xi|)$  equals one of following functions with corresponding additional conditions hold:*

Model	$f(x, t)$	Additional condition
Perturbed double phase	$t^p + a(x)t^q \log(e + t)$	$\alpha_a \geq q - p$
Triple phase	$t^p + a(x)t^q + b(x)t^r$	$\alpha_a \geq q - p$ & $\alpha_b \geq r - p$
Variable exponent double phase	$t^{p(x)} + a(x)t^{q(x)}$	$\alpha_a(x) \geq q(x) - p(x)$

Then every minimizer  $u \in W_{loc}^{1,1}(\Omega) \cap L^\infty(\Omega)$  of  $(\min F)$  satisfies  $u \in C_{loc}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$  independent of  $\|u\|_{L^\infty(\Omega)}$ .

**Remark 1.2** The previous corollary holds also with the weaker, but more difficult to check, assumption  $u \in W_{loc}^{1,1}(\Omega) \cap BMO(\Omega)$ . Without the additional a priori information  $u \in BMO(\Omega)$ , the additional conditions are

$$\alpha_a \geq \frac{n}{p}(q - p), \quad \alpha_b \geq \frac{n}{p}(r - p), \quad \text{and } \alpha_a(x) \geq \frac{n}{p(x)}(q(x) - p(x)).$$

These are stronger assumptions when  $p < n$  as expected, since if  $p \geq n$ , then  $W_{loc}^{1,p}(\Omega) \subset BMO_{loc}(\Omega)$ , so the a priori assumption  $u \in BMO(\Omega)$  actually contains no additional information.

**Remark 1.3** Our results also apply in the following cases with the same assumptions as in Corollary 1.1.

Model	$f(x, t)$
Perturbed variable exponent	$t^{p(x)} \log(e + t)$
Orlicz variable exponent	$\varphi(t)^{p(x)}$ or $\varphi(t^{p(x)})$
Double variable exponent	$t^{p(x)} + t^{q(x)}$
Degenerate double phase	$t^p + a(x)t^p \log(e + t)$

However, in these cases the a priori information does not give any improvement in the result. The reason is that for these energies, a calculation shows that (A1- $n$ ) holds if and only if the (A1) condition holds, see Sect. 3. In other words, for these cases we obtain a new proof of results previously obtained in [42].

On the other hand, a priori information does matter for the Orlicz double phase  $\varphi(t) + a(x)\psi(t)$ , where  $a \in C^{0,\lambda}$  and  $\psi/\varphi$  is almost increasing, but the conditions get a bit messy. The main condition is that for each  $\varepsilon > 0$  there exists  $\beta > 0$  such that

$$\lambda(r) \frac{\psi(r^{-1/(1+\varepsilon)})}{\varphi(r^{-1/(1+\varepsilon)})} \lesssim r^\beta.$$

The detailed calculations are left to the interested reader, cf. [41, Corollary 8.4].

We study regularity of weak solutions or minimizers  $u$  with the additional information that they belong to  $L^{s^*}$ ,  $BMO$ ,  $L^\infty$  or  $C^{0,\gamma}$ , and study sharp conditions on  $A$  or  $F$  corresponding to restrictions on  $u$ . See Definition 3.1 for these sharp conditions and Example 3.2 for their interpretation in the double phase case. The functions  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  from (div  $A$ ) and  $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  from (min  $F$ ) have quasi-isotropic  $(p, q)$ -growth structure, given in Definition 4.1. We briefly explain the strategy and structure of the paper.

In Sect. 3, we consider lower order regularity in cases of generalized Orlicz growth and a priori information. We prove  $C^{0,\alpha}$ -regularity for some  $\alpha \in (0, 1)$  and higher integrability for quasiminimizers (Theorems 3.12 and 3.14). These are based on Sobolev–Poincaré type inequalities with a priori information, which are obtained in Theorem 3.4 with Lemma 3.8.

In Sect. 5 we prove the main results, maximal regularity of weak solutions and minimizers for cases (ap2)–(ap4). We prove  $C^{0,\alpha}$ -regularity for every  $\alpha \in (0, 1)$  and  $C^{1,\alpha}$ -regularity for some  $\alpha \in (0, 1)$  assuming a priori  $C^{0,\gamma}$ -information (Theorems 5.3 and 5.11). Other cases follow as corollaries by the lower order regularity results. The crucial step of the proofs is approximating the original problem (div  $A$ ) and (min  $F$ ) with a suitable autonomous problem and obtaining a comparison estimate between solutions to the original problem and the autonomous problem. For the approximation we use tools from [42], but the comparison is achieved quite differently from our earlier papers. In this paper, we use harmonic approximation in Lemma 4.13 generalizing the double phase case from [8]. The main innovations are inventing assumptions and formulating results optimally to cover all special cases while also being sharp, see comments before Theorem 5.3 for details.

We start in Sect. 2 by recalling notation, definitions and basic results on generalized Orlicz spaces.

## 2 Preliminaries and notation

Throughout the paper we always assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with  $n \geq 2$ . For  $x_0 \in \mathbb{R}^n$  and  $r > 0$ ,  $B_r(x_0)$  is the open ball with center  $x_0$  and radius  $r$ . If its center is clear

or irrelevant, we write  $B_r = B_r(x_0)$ . The characteristic function  $\chi_E$  of  $E \subset \mathbb{R}^n$  is defined as  $\chi_E(x) = 1$  if  $x \in E$  and  $\chi_E(x) = 0$  if  $x \notin E$ .

Let  $f, g : E \rightarrow \mathbb{R}$  be measurable in  $E \subset \mathbb{R}^n$ . We denote the integral average of  $f$  over  $E$  with  $0 < |E| < \infty$  by  $(f)_E := \int_E f dx := \frac{1}{|E|} \int_E f dx$ . The gradient of  $f$  is denoted  $Df$ . If  $E \subset \mathbb{R}$ , then  $f$  is said to be almost increasing with constant  $L \geq 1$  if  $f(s) \leq Lf(t)$  whenever  $s \leq t$ . If  $L = 1$ ,  $f$  is increasing. Similarly, we define an almost decreasing or decreasing function. We write  $f \lesssim g, f \approx g$  and  $f \simeq g$  if there exists  $C \geq 1$  such that  $f(y) \leq Cg(y), C^{-1}f(y) \leq g(y) \leq Cf(y)$  and  $f(C^{-1}y) \leq g(y) \leq f(Cy)$  for all  $y \in E$ , respectively. We use  $c$  as a generic constant whose value may change between appearances.

A modulus of continuity  $\omega : [0, \infty) \rightarrow [0, \infty)$  is concave and increasing with  $\omega(0) = \lim_{r \rightarrow 0^+} \omega(r) = 0$ . We define the Hölder seminorm by

$$[u]_\gamma = [u]_{\gamma, \Omega} := \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma} \quad \text{and} \quad [u]_{\gamma, r} := \sup_{x \in \Omega} [u]_{\gamma, B_r(x) \cap \Omega}.$$

Vanishing Hölder continuity  $V C^{0, \gamma}(\Omega)$  means that  $u \in C^{0, \gamma}(\Omega)$  and  $\lim_{r \rightarrow 0^+} [u]_{\gamma, r} = 0$ . The spaces  $C^{0, \gamma(\cdot)}(\Omega)$  and  $C^{0, \log}(\Omega)$ , as well as their vanishing versions, are defined similarly with  $|x - y|^{\gamma(x)}$  and  $\log(e + \frac{1}{|x - y|})$  instead of  $|x - y|^\gamma$  in the denominator.

We refer to [33, Chapter 2] for the following definitions and properties.

**Definition 2.1** We define some conditions for  $\varphi : \Omega \times [0, \infty] \rightarrow [0, \infty)$  and  $\gamma \in \mathbb{R}$  related to regularity with respect to the second variable, which are supposed to hold for all  $x \in \Omega$  and a constant  $L \geq 1$  independent of  $x$ .

- (aInc) $_\gamma$   $t \mapsto \varphi(x, t)/t^\gamma$  is almost increasing on  $(0, \infty)$  with constant  $L$ .
- (Inc) $_\gamma$   $t \mapsto \varphi(x, t)/t^\gamma$  is increasing on  $(0, \infty)$ .
- (aDec) $_\gamma$   $t \mapsto \varphi(x, t)/t^\gamma$  is almost decreasing on  $(0, \infty)$  with constant  $L$ .
- (Dec) $_\gamma$   $t \mapsto \varphi(x, t)/t^\gamma$  is decreasing on  $(0, \infty)$ .
- (A0)  $L^{-1} \leq \varphi(x, 1) \leq L$ .

We write (aInc) or (aDec) if (aInc) $_\gamma$  or (aDec) $_\gamma$  holds for some  $\gamma > 1$ .

We can rewrite (aInc) $_p$  or (aDec) $_q$  with  $p, q > 0$  and constant  $L \geq 1$  as

$$\varphi(x, \lambda t) \leq L\lambda^p \varphi(x, t) \quad \text{and} \quad \varphi(x, \Lambda t) \leq L\Lambda^q \varphi(x, t), \quad \text{respectively,}$$

for all  $(x, t) \in \Omega \times [0, \infty)$  and  $0 \leq \lambda \leq 1 \leq \Lambda$ . From these inequalities one sees that (aInc) and (aDec) are equivalent to the  $\nabla_2$ - and  $\Delta_2$ -conditions, respectively. The definition of (A0) above differs slightly from [33] but the two definitions are equivalent when  $\varphi$  satisfies (aDec). If  $\varphi(x, \cdot) \in C^1((0, \infty))$ , then for  $0 < p < q$ ,

$$\varphi \text{ satisfies } (Inc)_p \text{ and } (Dec)_q \iff p \leq \frac{t\varphi'(x, t)}{\varphi(x, t)} \leq q \text{ for all } t \in (0, \infty).$$

Suppose  $\varphi, \psi : [0, \infty) \rightarrow [0, \infty)$  are increasing,  $\varphi$  satisfies (aInc) $_1$  and (aDec), and  $\psi$  satisfies (aDec) $_1$ . Then there exist a convex  $\tilde{\varphi}$  and a concave  $\tilde{\psi}$  such that  $\varphi \approx \tilde{\varphi}$  and  $\psi \approx \tilde{\psi}$  [33, Lemma 2.2.1]. Therefore, by Jensen's inequality for  $\tilde{\varphi}$  and  $\tilde{\psi}$ ,

$$\varphi \left( \int_\Omega |f| dx \right) \lesssim \int_\Omega \varphi(|f|) dx \quad \text{and} \quad \int_\Omega \psi(|f|) dx \lesssim \psi \left( \int_\Omega |f| dx \right)$$

for every  $f \in L^1(\Omega)$  with implicit constants depending on  $L$  from (aInc) $_1$  and (aDec) or (aDec) $_1$  (via the constants from the equivalence relation).

We next introduce classes of  $\Phi$ -functions and generalized Orlicz spaces following [33]. We are mainly interested in convex functions for minimization problems and related PDEs, but the class  $\Phi_w(\Omega)$  is very useful for approximating functionals.

**Definition 2.2** Let  $\varphi : \Omega \times [0, \infty] \rightarrow [0, \infty)$ . Assume  $x \mapsto \varphi(x, |f(x)|)$  is measurable for every measurable function  $f$  on  $\Omega$ ,  $t \mapsto \varphi(x, t)$  is increasing for every  $x \in \Omega$ , and  $\varphi(x, 0) = \lim_{t \rightarrow 0^+} \varphi(x, t) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$  for every  $x \in \Omega$ . Then  $\varphi$  is called a

- (1)  $\Phi$ -function, denoted  $\varphi \in \Phi_w(\Omega)$ , if it satisfies (aInc)<sub>1</sub>;
- (2) convex  $\Phi$ -function, denoted  $\varphi \in \Phi_c(\Omega)$ , if  $t \mapsto \varphi(x, t)$  is left-continuous and convex for every  $x \in \Omega$ .

If  $\bar{\varphi}$  is independent of  $x$  and  $\varphi(x, t) := \bar{\varphi}(t)$  satisfies  $\varphi \in \Phi_w(\Omega)$  or  $\varphi \in \Phi_c(\Omega)$ , we write  $\bar{\varphi} \in \Phi_w$  or  $\bar{\varphi} \in \Phi_c$ .

Note that  $\Phi_c(\Omega) \subset \Phi_w(\Omega)$  since convexity implies (Inc)<sub>1</sub>. For  $\varphi, \psi \in \Phi_w(\Omega)$  the relation  $\simeq$  is weaker than  $\approx$ , but they are equivalent if  $\varphi$  and  $\psi$  satisfy (aDec). We write

$$\varphi_{B_r}^+(t) := \sup_{x \in B_r \cap \Omega} \varphi(x, t) \quad \text{and} \quad \varphi_{B_r}^-(t) := \inf_{x \in B_r \cap \Omega} \varphi(x, t).$$

The (left-continuous) inverse function with respect to  $t$  is defined by

$$\varphi^{-1}(x, t) := \inf\{\tau \geq 0 : \varphi(x, \tau) \geq t\}.$$

If  $\varphi$  is strictly increasing and continuous in  $t$ , then this is just the normal inverse. We define the conjugate function of  $\varphi \in \Phi_w(\Omega)$  by

$$\varphi^*(x, t) := \sup_{s \geq 0} (st - \varphi(x, s)).$$

The definition directly implies Young’s inequality

$$ts \leq \varphi(x, t) + \varphi^*(x, s) \quad \text{for all } s, t \geq 0.$$

If  $\varphi$  satisfies (aInc) <sub>$p$</sub>  or (aDec) <sub>$q$</sub>  for some  $p, q > 1$ , then  $\varphi^*$  satisfies (aDec) <sub>$p'$</sub>  or (aInc) <sub>$q'$</sub> , respectively; the prime denotes the Hölder conjugate,  $p' = \frac{p}{p-1}$ . We also note that  $(\varphi^*)^* = \varphi$  if  $\varphi \in \Phi_c(\Omega)$  by [24, Theorem 2.2.6].

If  $\varphi \in \Phi_c(\Omega)$ , then there exists an increasing and right-continuous  $\varphi' : \Omega \times [0, \infty) \rightarrow [0, \infty)$  such that

$$\varphi(x, t) = \int_0^t \varphi'(x, s) ds.$$

We collect some results about this (right-)derivative  $\varphi'$ .

**Proposition 2.3** (Proposition 3.6, [41]) Let  $\gamma > 0$  and  $\varphi \in \Phi_c(\Omega)$ .

- (1) If  $\varphi'$  satisfies (Inc) <sub>$\gamma$</sub> , (Dec) <sub>$\gamma$</sub> , (aInc) <sub>$\gamma$</sub>  or (aDec) <sub>$\gamma$</sub> , then  $\varphi$  satisfies (Inc) <sub>$\gamma+1$</sub> , (Dec) <sub>$\gamma+1$</sub> , (aInc) <sub>$\gamma+1$</sub>  or (aDec) <sub>$\gamma+1$</sub> , respectively, with the same constant  $L \geq 1$ .
- (2) If  $\varphi$  satisfies (aDec) <sub>$\gamma$</sub> , then  $(2^{\gamma+1}L)^{-1}t\varphi'(x, t) \leq \varphi(x, t) \leq t\varphi'(x, t)$ .
- (3) If  $\varphi'$  satisfies (A0) and (aDec) <sub>$\gamma$</sub>  with constant  $L \geq 1$ , then  $\varphi$  also satisfies (A0), with constant depending on  $L$  and  $\gamma$ .
- (4)  $\varphi^*(x, \varphi'(x, t)) \leq t\varphi'(x, t)$ .

Let  $L^0(\Omega)$  be the set of the measurable functions on  $\Omega$ . For  $\varphi \in \Phi_w(\Omega)$ , the *generalized Orlicz space* (also known as the *Musielak–Orlicz space*) is defined as

$$L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \|f\|_{L^\varphi(\Omega)} < \infty \right\},$$

with the (Luxemburg) norm

$$\|f\|_{L^\varphi(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1 \right\}, \quad \text{where } \varrho_\varphi(f) := \int_\Omega \varphi(x, |f|) dx.$$

We denote by  $W^{1,\varphi}(\Omega)$  the set of functions  $f \in W^{1,1}(\Omega)$  with  $\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^\varphi(\Omega)} + \| |Df| \|_{L^\varphi(\Omega)} < \infty$ . Note that if  $\varphi$  satisfies (aDec), then  $f \in L^\varphi(\Omega)$  if and only if  $\varrho_\varphi(f) < \infty$ , and if  $\varphi$  satisfies (A0), (aInc) and (aDec), then  $L^\varphi(\Omega)$  and  $W^{1,\varphi}(\Omega)$  are reflexive Banach spaces. We denote by  $W_0^{1,\varphi}(\Omega)$  the closure of  $C_0^\infty(\Omega)$  in  $W^{1,\varphi}(\Omega)$ . For more information about generalized Orlicz and Orlicz–Sobolev spaces, we refer to the monographs [14, 33] and also [24, Chapter 2].

### 3 Lower regularity with a priori assumptions

#### Continuity assumptions

The condition (A1), introduced in [39] (see also [44]), is a ‘‘almost continuity’’ assumption, which allows the function to jump, but not too much. It implies the Hölder continuity of solutions and (quasi)minimizers [11, 36, 37]. For higher regularity, we introduced in [41] a ‘‘vanishing (A1)’’ condition, denoted (VA1), and a weak vanishing version, (wVA1) and generalized them to the quasi-isotropic situation in [42]. The anisotropic condition was further studied in [13, 40]. These previous studies applied to the ‘‘natural’’ energy assumption  $u \in W^{1,\varphi}(\Omega)$ , called Case (ap1) in the introduction. The (A1- $n$ ) and (A1- $\psi$ ) conditions for a priori energy assumptions were developed in [36] and [11] for functions in  $L^\infty$  and  $W^{1,\psi}$ , respectively. Here we generalize and unify all the conditions for a priori information; the most important one for this article is (VA1- $s$ ).

**Definition 3.1** Let  $M, N \in \mathbb{N}$ ,  $G : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^N$ ,  $\psi : \Omega \times \mathbb{R}^M \rightarrow [0, \infty)$ ,  $L_\omega > 0$ ,  $r \in (0, 1]$  and  $\omega : [0, 1] \rightarrow [0, L_\omega]$ . We consider the claim

$$|G(x, \xi) - G(y, \xi)| \leq \omega(r)(|G(y, \xi)| + 1) \quad \text{when } \psi(y, \xi) \in [0, |B_r|^{-1}]$$

for all  $x, y \in B_r \cap \Omega$  and  $\xi \in \mathbb{R}^M$ . We say that  $G$  satisfies:

- (A1- $\psi$ ) if there exists  $L_\omega$  such that the claim holds with  $\omega \equiv L_\omega$ .
- (VA1- $\psi$ ) if there exists  $L_\omega$  and a modulus of continuity  $\omega$  such that the claim holds.
- (wVA1- $\psi$ ) if it satisfies (VA1- $\psi^{1+\varepsilon}$ ) for every  $\varepsilon > 0$ , with possibly different functions  $\omega_\varepsilon$  but a common  $L_\omega$  independent of  $\varepsilon$ .

When  $\psi(x, \xi) = |\xi|^s$  with  $s > 0$  we use the abbreviations (A1- $s$ ), (VA1- $s$ ) and (wVA1- $s$ ) and in the case  $\psi = |G|$  we write (A1), (VA1) and (wVA1). We also use the definition for  $\psi : \Omega \times [0, \infty) \rightarrow [0, \infty)$  with the understanding that  $\psi(x, \xi) = \psi(x, |\xi|)$ .

It can be easily seen that (VA1- $s$ )  $\implies$  (wVA1- $s$ )  $\implies$  (A1- $s$ ) and (VA1- $s'$ )  $\implies$  (VA1- $s$ ) if  $s' \leq s$ , similarly for (wVA1- $s$ ) and (A1- $s$ ). In the case  $M = N = 1$ , these conditions are somewhat differently formulated than in earlier papers, but we showed in [42] that the formulations are equivalent to previous versions under natural assumptions on  $G$ .

**Table 1** Examples of sufficient conditions in special cases

Model and function	(A1-s)	(wVA1-s)	(VA1-s)
Variable exponent $t^{p(x)}$	$p \in C^{\log}$	$p \in VC^{\log}$	$p \in VC^{\log}$
Orlicz variable exponent $\psi(t)^{p(x)}$	$p \in C^{\log}$	$p \in VC^{\log}$	$p \in VC^{\log}$
Double phase $t^p + a(x)t^q, \alpha := \frac{n}{s}(q - p)$	$a \in C^\alpha$	$a \in C^\alpha$	$a \in VC^\alpha$
Double phase variable exponent $t^{p(x)} + a(x)t^{q(x)}, p, q \in C^{\log}, \alpha := \frac{n}{s}(q - p)$	$a \in C^{\alpha(\cdot)}$	$a \in C^{\alpha(\cdot)}$	$a \in VC^{\alpha(\cdot)}$

The next example shows the relevance of the parameter  $s$  in (A1-s) in the double phase case. Similar relations for other cases are summarized in Table 1.

**Example 3.2** Consider two double phase energies for  $1 < p \leq q$  and  $a : \Omega \rightarrow [0, L]$ .

- Let  $\varphi_1(x, t) := t^p + a(x)t^q, a \in VC^{0,\alpha}(\Omega)$  for some  $\alpha \in (0, 1]$ .  
 If  $q - p \leq \frac{s}{n}\alpha$ , then  $\varphi_1$  satisfies (VA1-s) with  $\omega$  proportional to the modulus of continuity of  $a$ .
- Let  $\varphi_2(x, t) := t^p + a(x)^\alpha t^q$ , where  $\alpha > 0$  and  $a \in C^{0,1}(\Omega)$ .  
 If  $q - p \leq \frac{s}{n}\alpha$ , then  $\varphi_2$  satisfies (A1-s) and (wVA1-s) with  $\omega_\varepsilon(r) = cr^{\{\alpha - \frac{n(q-p)}{(1+\varepsilon)s}\} \min\{1, \frac{1}{\alpha}\}}$ ;  
 If  $q - p < \frac{s}{n}\alpha$ , then  $\varphi_2$  satisfies (VA1-s) with  $\omega(r) = cr^{\{\alpha - \frac{n(q-p)}{s}\} \min\{1, \frac{1}{\alpha}\}}$ .

Note that these conditions can hold for  $\frac{q}{p}$  arbitrarily large. We show the second case only since the first case can be obtained in the same way as the second case with  $\alpha < 1$ . We will use the elementary inequality

$$|a^\alpha - b^\alpha| \leq \begin{cases} |a - b|^\alpha, & 0 < \alpha \leq 1, \\ c_\alpha \delta^{1-\alpha} |a - b|^\alpha + \delta b^\alpha, & \alpha > 1, \end{cases}$$

which holds for any  $a, b \geq 0$  and  $\delta \in (0, 1]$ ; here  $c_\alpha > 0$  is a constant depending on  $\alpha$ . The second inequality ( $\alpha > 1$ ) follows from Young’s inequality applied to the right-hand side of  $b^\alpha - a^\alpha \leq \alpha b^{\alpha-1}(b - a)$  when  $b \geq a \geq 0$ .

Suppose that  $q - p \leq \frac{s}{n}\alpha$  and let  $x, y \in B_r$  with  $r \in (0, 1]$  and  $t \in (0, |B_r|^{-\frac{1}{s(1+\varepsilon)}}]$  with  $\varepsilon \geq 0$ . Applying the preceding inequality with  $a = a(x)$  and  $b = a(y)$ , we obtain that

$$|\varphi_2(x, t) - \varphi_2(y, t)| \lesssim r^\alpha t^q = r^\alpha t^{q-p} t^p \lesssim r^{\alpha - \frac{n(q-p)}{(1+\varepsilon)s}} \varphi_2(y, t)$$

when  $0 < \alpha \leq 1$ ; in the case  $\alpha > 1$ , we choose  $\delta := r^{\{\alpha - \frac{n(q-p)}{(1+\varepsilon)s}\} \frac{1}{\alpha}}$  and find that

$$\begin{aligned} |\varphi_2(x, t) - \varphi_2(y, t)| &\lesssim (\delta^{1-\alpha} r^\alpha + \delta a(y)^\alpha) t^q \\ &\lesssim \delta^{1-\alpha} r^{\alpha - \frac{n(q-p)}{(1+\varepsilon)s}} t^p + \delta a(y)^\alpha t^q = r^{\{\alpha - \frac{n(q-p)}{(1+\varepsilon)s}\} \frac{1}{\alpha}} \varphi_2(y, t). \end{aligned}$$

These inequalities imply the desired (A1-s), (wVA1-s) and (VA1-s)-conditions.

Let us show how to use the smallness of  $\omega$  to obtain the inequality from (VA1-s) for a slightly larger range. Intuitively, we shift some power from the coefficient to the range. In this proof it is important that the range of  $\xi$  in the condition is independent of  $x$ , so the result does not generalize to (VA1- $\psi$ ) easily, unless  $\psi(x, t) = \psi(t)$ .

**Proposition 3.3** *Let  $G : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^N$  satisfy (VA1-s),  $\theta \in [0, 1]$  and  $r \in (0, 1]$ . If  $\omega(r)^{1-\theta} \leq \frac{1}{3}$ , then for every  $x, y \in B_r \cap \Omega$ ,*

$$|G(x, \xi) - G(y, \xi)| \leq \omega(r)^\theta (|G(y, \xi)| + 1) \text{ when } 3^n \omega(r)^{n(1-\theta)} |\xi|^s \in [0, |B_r|^{-1}].$$

**Proof** Note by the concavity of log that  $t \log 2 \leq \log(1 + t) \leq t$  when  $t \in [0, 1]$ , and set

$$k := \left\lfloor \frac{\log(1 + \omega(r)^\theta)}{\log(1 + \omega(r))} \right\rfloor \geq \lfloor \omega(r)^{\theta-1} \log 2 \rfloor \geq \frac{\log 2}{2} \omega(r)^{\theta-1} > \frac{1}{3} \omega(r)^{\theta-1} \geq 1,$$

where  $\lfloor \tau \rfloor$  is the largest integer less than or equal to  $\tau \in \mathbb{R}$ . Suppose  $x, y \in B_r \cap \Omega$  and  $3^n \omega(r)^{n(1-\theta)} |\xi|^s \leq |B_r|^{-1}$ . Then  $|\xi|^s \leq k^n |B_r|^{-1}$  since  $1 \leq (3k)^n \omega(r)^{n(1-\theta)}$ . We split the segment  $[x, y]$  into  $k$  equally long subsegments  $[x_i, x_{i+1}]$  with  $x_0 = x$  and  $x_k = y$  so that  $x_i, x_{i+1} \in B_{r/k}$ . Since  $|\xi|^s \leq |B_{r/k}|^{-1}$ , we can use (VA1-s) to estimate

$$|G(x_i, \xi)| + 1 \leq (1 + \omega(r))(|G(x_{i+1}, \xi)| + 1) \leq \dots \leq (1 + \omega(r))^{k-i} (|G(y, \xi)| + 1).$$

We use this estimate with the triangle inequality and (VA1-s):

$$\begin{aligned} |G(x, \xi) - G(y, \xi)| &\leq \sum_{i=0}^{k-1} |G(x_{i+1}, \xi) - G(x_i, \xi)| \leq \omega(r) \sum_{i=1}^k (|G(x_i, \xi)| + 1) \\ &\leq \omega(r) \sum_{i=1}^k (1 + \omega(r))^{k-i} (|G(y, \xi)| + 1) \\ &= [(1 + \omega(r))^k - 1] (|G(y, \xi)| + 1). \end{aligned}$$

This gives the desired estimate, since by the definition of  $k$ ,

$$(1 + \omega(r))^k \leq (1 + \omega(r))^{\frac{\log(1+\omega(r)^\theta)}{\log(1+\omega(r))}} = 1 + \omega(r)^\theta.$$

□

### Sobolev–Poincaré inequality

We derive a modular Sobolev–Poincaré-type inequality in generalized Orlicz spaces assuming a priori information. We first state the inequality with an abstract condition, which is explored further in Lemma 3.8 and Example 3.9. The example shows that the conditions in Lemma 3.8 are essentially sharp for the Sobolev–Poincaré inequality, at least when  $s \leq n$ . This approach is inspired by [11].

**Theorem 3.4** (Sobolev–Poincaré inequality) *Let  $\varphi \in \Phi_w(B_r)$  satisfy (A0), (aInc)<sub>p</sub> and (aDec)<sub>q</sub> with  $1 \leq p \leq q$ , and let  $u \in W^{1,1}(B_r)$ . If*

$$\int_{B_r} \left( \frac{\varphi(x, v)}{\varphi_{B_r}^-(v) + 1} \right)^\theta dx \leq b_0 \text{ for } v := \frac{|u - (u)_{B_r}|}{r} \tag{3.5}$$

and some  $b_0, \theta > 0$ , then

$$\left( \int_{B_r} \varphi(x, v)^{\theta_0} dx \right)^{\frac{1}{\theta_0}} \leq c \int_{B_r} \varphi_{B_r}^- (|Du|) dx + c$$

for  $\frac{1}{\theta_0} = 1 - \min\{\frac{p}{n}, \kappa\} + \frac{1}{\theta}$  with any  $\kappa \in (0, 1)$  and some  $c = c(n, p, q, L, \kappa, b_0) > 0$ .

**Remark 3.6** In the previous theorem we can choose  $\theta_0 > 1$  if and only if  $\theta > \max\{1, \frac{n}{p}\}$ . The choice  $\theta_0 = 1$  is additionally possible when  $\theta = \frac{n}{p} > 1$ . These are the most important cases, but the theorem allows also for  $\theta_0 \in (0, 1)$  in cases with small  $\theta$ .

**Remark 3.7** Let  $1 < p \leq q$ . Suppose that  $\varphi$  satisfies (3.5) with  $\theta$  such that  $\theta_0 > 1$ . Then so does  $\varphi^{1/\tau}$  with  $\theta' := \theta\tau$  and  $\tau \in (1, p)$ . Theorem 3.4 for  $\varphi^{1/\tau}$  implies that

$$\left( \int_{B_r} \varphi(x, v)^{\frac{\theta'_0}{\tau}} dx \right)^{\frac{\tau}{\theta'_0}} \leq c \left( \int_{B_r} \varphi(x, |Du|)^{\frac{1}{\tau}} dx \right)^{\tau} + c,$$

where  $\frac{1}{\theta'_0} = 1 - \min\{\frac{p}{n}, \kappa\} + \frac{1}{\theta\tau}$ . Note that  $\theta'_0 \rightarrow \theta_0 > 1$  when  $\tau \rightarrow 1$  so we can choose  $\tau$  with  $\frac{\theta'_0}{\tau} > 1$ . Thus there exists  $\theta_1 > 1$  depending  $n, p$  and  $\theta$  such that

$$\left( \int_{B_r} \varphi(x, v)^{\theta_1} dx \right)^{\frac{1}{\theta_1}} \leq c \left( \int_{B_r} \varphi(x, |Du|)^{\frac{1}{\theta_1}} dx \right)^{\theta_1} + c.$$

**Proof of Theorem 3.4** To obtain a differentiable function, we define  $\psi \in \Phi_c$  by

$$\psi(t) := \int_0^t \sup_{\sigma \in (0, \tau]} \frac{\varphi_{B_r}^-(\sigma)}{\sigma} d\tau.$$

From (aInc)<sub>1</sub> of  $\varphi$  we see that  $\frac{\varphi_{B_r}^-(t)}{t} \leq \psi'(t) \leq L \frac{\varphi_{B_r}^-(t)}{t}$ ; with (aInc)<sub>1</sub> and (aDec)<sub>q</sub> we conclude that  $\varphi_{B_r}^- \approx \psi(t)$ . Choose  $s := \min\{p, \kappa n\} \in [1, n)$  and  $\theta_0 \in (0, \theta)$  with  $\frac{s}{n} = 1 - \frac{1}{\theta_0} + \frac{1}{\theta}$ . Note that  $\frac{s}{\theta_0 s} = ((1 - \frac{s}{n})\theta_0)^{-1} = (1 - \frac{\theta_0}{\theta})^{-1} > 1$ . By Hölder’s inequality with exponents  $\frac{s^*}{\theta_0 s}$  and  $(\frac{s^*}{\theta_0 s})' = \frac{\theta}{\theta_0}$  and the assumption (3.5),

$$\left( \int_{B_r} \varphi(x, v)^{\theta_0} dx \right)^{\frac{1}{\theta_0}} \lesssim b_0^{\frac{1}{\theta}} \left( \int_{B_r} \psi(v)^{\frac{s^*}{s}} dx + 1 \right)^{\frac{s}{s^*}}.$$

We use the Sobolev–Poincaré inequality in  $L^s$  and  $\psi'(t) \approx \psi(t)/t$  to conclude that

$$\begin{aligned} \left( \int_{B_r} |\psi(v)^{\frac{1}{s}} - (\psi(v)^{\frac{1}{s}})_{B_r}|^{s^*} dx \right)^{\frac{s}{s^*}} &\lesssim r^s \int_{B_r} |\nabla(\psi(v)^{\frac{1}{s}})|^s dx \\ &\approx r^s \int_{B_r} \psi(v)^{1-s} \psi'(v)^s |\nabla v|^s dx \\ &\approx \int_{B_r} \frac{\psi(v)}{v^s} |\nabla u|^s dx. \end{aligned}$$

Since  $\psi$  satisfies (aInc)<sub>p</sub> and  $s \leq p$ ,  $\psi_s(t) := \psi(t^{1/s})$  satisfies (aInc)<sub>1</sub>. Therefore Young’s inequality for  $\psi_s$  and  $\psi_s^*(\frac{\psi_s(t)}{Lt}) \leq \psi_s(t)$  [35, Lemma 3.1] with  $t := v^s$  give  $\frac{\psi(v)}{v^s} |\nabla u|^s \lesssim \psi(v) + \psi(|\nabla u|)$ . Continuing the previous estimate with the  $L^{s^*}$ -triangle inequality and  $\psi \approx \varphi_{B_r}^-$ , we find that

$$\left( \int_{B_r} \psi(v)^{\frac{s^*}{s}} dx \right)^{\frac{s}{s^*}} \lesssim \int_{B_r} \varphi_{B_r}^- (|\nabla u|) dx + \int_{B_r} \psi(v) dx + (\psi(v)^{\frac{1}{s}})_{B_r}^s.$$

By Hölder’s inequality,  $(\psi(v)^{\frac{1}{s}})_{B_r}^s \leq (\psi(v))_{B_r}$  and by the modular Poincaré inequality in the Orlicz space  $L^\psi$  [33, Corollary 7.4.1],  $(\psi(v))_{B_r}$  can be estimated by the first term on the

right-hand side. Combined with the inequality from the previous paragraph, this gives the claim.  $\square$

Let us derive some sufficient conditions for the assumption of the previous theorem by complementing [11, Proposition 4.2]. Also the cases  $u \in L^{\psi^\#}(B_r)$  and  $u \in W^{1,\psi}(B_r)$  for  $\psi \in \Phi_w(B_r)$  and (A1- $\psi$ ) are covered by [11], and could likewise be considered here. We define the *bounded mean oscillation* semi-norm as

$$[u]_{BMO(B_r)} := \sup_{B_\rho \subset B_r} \int_{B_\rho} |u - (u)_{B_\rho}| dx.$$

Note that in case (3) of the following lemma we need  $s > n(1 - \frac{p}{q})$  in order that  $\theta_0 > 1$  in Theorem 3.4, cf. Remark 3.6.

**Lemma 3.8** *Let  $\varphi \in \Phi_w(B_r)$  satisfy (A0), (aInc) $_p$ , (aDec) $_q$  and (A1- $s$ ) with  $1 \leq p \leq q$  and let  $u \in L^1(B_r)$ . Assume one of the following holds:*

- (1)  $s > n$  and  $[u]_{\gamma, B_r} \leq b$  with  $\gamma := 1 - \frac{n}{s}$ .
- (2)  $s = n$  and  $[u]_{BMO(B_r)} \leq b$ .
- (3)  $s \in [1, n)$  and  $\|u\|_{L^{s^*}(B_r)} \leq b$ .

Then (3.5) holds for any  $\theta > 0$  in Cases (1)–(2) and for  $\theta = \frac{s^*}{q-p}$  in Case (3). The constant  $b_0$  depends only on  $n, p, q, L, L_\omega$  and  $b$ .

**Proof** If  $[u]_\gamma \leq b$ , then  $v \leq 2|B_1|^{\frac{1}{n}} b |B_r|^{-\frac{\gamma-1}{n}}$  and  $\varphi_{B_r}^+(v) \lesssim \varphi_{B_r}^-(v) + 1$  by (A1- $s$ ), so the claim holds in Case (1). For the same reason and (A0), the integrand in (3.5) in Case (2) is bounded at points with  $v \leq \max\{|B_r|^{-1/n}, 1\}$ . On the other hand, at points with  $v > \max\{|B_r|^{-1/n}, 1\}$  we estimate, by (A0), (aInc) $_p$ , (aDec) $_q$  and (A1- $n$ ),

$$\frac{\varphi(x, v)}{\varphi_{B_r}^-(v) + 1} \approx \frac{\varphi(x, v)}{\varphi_{B_r}^-(v)} \lesssim \left(\frac{v}{|B_r|^{-1/n}}\right)^{q-p} \frac{\varphi(x, |B_r|^{-1/n})}{\varphi_{B_r}^-(|B_r|^{-1/n})} \approx \left(\frac{v}{r^{-1}}\right)^{q-p} = |u - (u)_{B_r}|^{q-p}.$$

We obtain for any exponent  $\theta > 0$  that

$$\int_{B_r} \left(\frac{\varphi(x, v)}{\varphi_{B_r}^-(v) + 1}\right)^\theta dx \lesssim \int_{B_r} |u - (u)_{B_r}|^{\theta(q-p)} dx + 1 \lesssim [u]_{BMO(B_r)}^{\theta(q-p)} + 1,$$

where in the last inequality we use the well-known reverse Hölder type inequality for mean oscillations in  $L^{\theta(q-p)}$ -space and  $BMO$  (cf. [26, Lemma A.1]). Case (3) was proved in [11, Proposition 4.2].  $\square$

The estimates for the Sobolev–Poincaré inequality may seem crude, but the following example shows that the end result is sharp, i.e. the claim is false if (A1- $s$ ) is replaced by (A1- $s'$ ) for any  $s' < s$ . See also [11, Section 5] for a one-dimensional example.

**Example 3.9** Let  $n = 2$  and denote the quadrants by  $Q_k \subset \mathbb{R}^2, k \in \{1, 2, 3, 4\}$ . Let  $\eta : [0, 4] \rightarrow [0, 1]$  be the piecewise linear, 3-Lipschitz function with  $\eta_{[\frac{1}{3}, \frac{2}{3}]} = \eta_{[\frac{7}{3}, \frac{8}{3}]} = 1$  and  $\eta_{[1, 2]} = \eta_{[3, 4]} = 0$ . We define  $a : \mathbb{R}^2 \rightarrow [0, \infty)$  in polar coordinates as  $a(r, \theta) := r^\alpha \eta(\frac{2}{\pi}\theta)$ . Thus  $a$  equals 0 in  $Q_2$  and  $Q_4$  and  $a(x) = |x|^\alpha$  in the sectors with  $\frac{\pi}{6} < \theta < \frac{\pi}{3}$  in  $Q_1$  and with  $\frac{7\pi}{6} < \theta < \frac{4\pi}{3}$  in  $Q_3$ . Consider the double phase functional  $H(x, t) = t^p + a(x)t^q$  with  $p < 2$  and the function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  which equals 1 in  $Q_1, -1$  in  $Q_3$  and is linear in the polar coordinate  $\theta$  in  $Q_2$  and  $Q_4$ . By symmetry,  $u_{B_r} = 0$  for every ball  $B_r$  centered at the

origin and  $v := \frac{1}{r}|u - (u)_{B_r}| = \frac{1}{r}$  in the sectors in  $Q_1$  and  $Q_3$ . The derivative of  $u$  equals zero in  $Q_1$  and  $Q_3$ ; in the other quadrants the radial derivative is zero, and in the tangential derivative equals  $\frac{4}{\pi r}$ . For a constant  $k > 0$  we estimate, based on the sectors in  $Q_1$  and  $Q_3$ ,

$$\int_{B_r} H(x, kv) dx \geq \frac{1}{3r^2} \int_0^r s^{1+\alpha} \left(\frac{k}{r}\right)^q ds = \frac{1}{3(2+\alpha)} r^{\alpha-q} k^q$$

and, since the support of the derivative is  $Q_2 \cup Q_4$  where  $a = 0$ ,

$$\int_{B_r} H(x, k|\nabla u|) dx = \frac{1}{r^2} \int_0^r s \left(\frac{4k}{\pi s}\right)^p ds = \frac{1}{2-p} \left(\frac{4}{\pi}\right)^p r^{-p} k^p.$$

If the modular Poincaré inequality from Theorem 3.4 holds with  $\theta_0 = 1$  (the weakest relevant case), then

$$r^{\alpha-q} k^q \leq cr^{-p} k^p + c \quad \text{so that} \quad r^{\alpha-(q-p)} k^{q-p} \leq c + cr^p k^{-p}.$$

Suppose that we want the constant in the inequality to depend on the  $L^{s^*}$ -norm of  $ku$ . We calculate  $\|ku\|_{L^{s^*}(B_r)} = ckr^{2/s^*}$ . Thus  $k = cr^{-2/s^*}$  and so

$$r^{\alpha-(q-p)} k^{q-p} \approx r^{\alpha-(\frac{2}{s^*}+1)(q-p)} = r^{\alpha-\frac{2}{s}(q-p)}.$$

This remains bounded as  $r \rightarrow 0$  when  $\alpha \geq \frac{2}{s}(q-p)$  which is exactly the (A1-s) condition when  $n = 2$  and shows the sharpness of Case (3). If  $s = n = 2$ , this shows the sharpness of Case (2), even if we allow the constant to depend on the  $L^\infty$ -norm. Similarly, we see that if the constant is allowed to depend on  $\rho_H(|\nabla u|)$ , then  $\alpha \geq \frac{2}{p}(q-p)$  which is (A1). Unfortunately, Case (1) is not covered, since the counter-example is discontinuous.

### Quasiminimizers

In this subsection, we derive regularity results for quasiminimizers with a priori information. Let  $\varphi \in \Phi_w(\Omega)$ . We say that  $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$  is a (local) quasiminimizer if there exists  $Q \geq 1$  such that

$$\int_{\text{supp}(u-v)} \varphi(x, |Du|) dx \leq Q \int_{\text{supp}(u-v)} \varphi(x, |Dv|) dx$$

for every  $v \in W_{\text{loc}}^{1,\varphi}(\Omega)$  with  $\text{supp}(u-v) \Subset \Omega$ . Quasiminimizers of energy functionals with generalized Orlicz growth have been studied e.g. in [11, 12, 36–38]. If  $\varphi$  satisfies (aDec) $_q$ , then the quasiminimizer  $u$  satisfies the Caccioppoli inequality

$$\int_{B_r} \varphi(x, |Du|) dx \leq c \int_{B_{2r}} \varphi\left(x, \frac{|u - (u)_{B_{2r}}|}{r}\right) dx, \tag{3.10}$$

for some  $c = c(n, q, L, Q) \geq 1$  and every  $B_{2r} \Subset \Omega$ , see [36, Lemma 4.6].

If  $\varphi$  satisfies (A1-n), then a bounded quasiminimizer satisfies a Harnack-type inequality and so is locally Hölder continuous [36, Theorem 4.1]. The main ingredients of the proof are the Caccioppoli estimate (3.10) and the Sobolev–Poincaré inequality (Theorem 3.4). In Lemma 3.8, we derived several sufficient conditions for the Sobolev–Poincaré inequality. Therefore, we obtain the Hölder continuity under these conditions from almost the same proof as [36, Theorem 4.1]. We start with local boundedness of quasiminimizers. The proof is exactly the same as [36, Proposition 5.5] and is hence omitted.

**Lemma 3.11** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (A0), (aInc) $_p$ , (aDec) $_q$  and (A1-s) with  $1 < p \leq q$ , and  $s \in (1, n]$ . Assume that  $u \in W_{loc}^{1,\varphi}(\Omega)$  is a quasiminimizer and  $u \in BMO(\Omega)$  with  $s = n$ , or  $u \in L^{s^*}(\Omega)$  with  $s \in (n(1 - \frac{p}{q}), n)$ . Then  $u \in L_{loc}^\infty(\Omega)$ .*

**Theorem 3.12** *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (A0), (aInc) $_p$ , (aDec) $_q$  and (A1-s) with  $1 < p \leq q$ , and  $s \in (1, n]$ . Assume that  $u \in W_{loc}^{1,\varphi}(\Omega)$  is a quasiminimizer and one of the following holds:*

- (1)  $s = n$  and  $u \in BMO(\Omega)$ .
- (2)  $s \in (n(1 - \frac{p}{q}), n)$  and  $u \in L^{s^*}(\Omega)$ .

*For every  $\Omega' \Subset \Omega$  there exists  $\gamma \in (0, 1)$  depending only on  $n, p, q, L, L_\omega, Q$  and  $\Omega'$  such that  $u \in C^{0,\gamma}(\Omega')$ .*

**Proof** Assume first that  $\varphi$  satisfies (A1-n) and  $u \in W_{loc}^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$ . Then local Hölder continuity follows directly from the Harnack inequality in [36, Theorem 4.1]. We consider then assumptions (1) and (2) and note that Lemma 3.11 implies that  $u \in L_{loc}^\infty(\Omega)$ . Furthermore, (A1-s) implies (A1-n) when  $s < n$ . Therefore, we obtain the local Hölder continuity from the result for bounded solutions. □

We end the section with a higher integrability result for Hölder continuous quasiminimizers. We first observe that if  $u \in C^{0,\gamma}(B_{2r})$  for some  $\gamma \in (0, 1)$  and  $\varphi$  satisfies (A1- $\frac{n}{1-\gamma}$ ), then by Jensen’s inequality and the Caccioppoli inequality (3.10)

$$\begin{aligned} \varphi_{B_{2r}}^- \left( \int_{B_r} |Du| dx \right) &\lesssim \int_{B_r} \varphi_{B_{2r}}^- (|Du|) dx \lesssim \int_{B_{2r}} \varphi \left( x, \frac{|u - (u)_{B_{2r}}|}{r} \right) dx \\ &\leq \int_{B_{2r}} \varphi(x, [u]_\gamma (2r)^{\gamma-1}) dx \lesssim ([u]_\gamma^p + [u]_\gamma^q) \int_{B_{2r}} \varphi(x, r^{\gamma-1}) dx \\ &\lesssim ([u]_\gamma^p + [u]_\gamma^q) (\varphi_{B_{2r}}^- (r^{\gamma-1}) + 1) \end{aligned}$$

which implies

$$\int_{B_r} |Du| dx \leq c \left( [u]_\gamma^{\frac{p}{q}} + [u]_\gamma^{\frac{p}{q}} \right) (r^{\gamma-1} + 1). \tag{3.13}$$

Here  $c$  depends on  $n, p, q, L, L_\omega$  and  $Q$ , and is independent of  $\gamma$ .

**Theorem 3.14** (Reverse Hölder inequality) *Let  $\varphi \in \Phi_w(\Omega)$  satisfy (A0), (aInc) $_p$ , (aDec) $_q$  and (A1- $\frac{n}{1-\gamma}$ ) with  $1 < p \leq q$  and  $\gamma \in (0, 1)$ . If  $u \in W_{loc}^{1,\varphi}(\Omega) \cap C^{0,\gamma}(B_{4r})$  is a quasiminimizer in  $B_{4r} \Subset \Omega$ , then  $|Du| \in L^{\varphi^{1+\sigma}}(B_r)$  for some  $\sigma > 0$  with the estimate*

$$\left( \int_{B_r} \varphi(x, |Du|)^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \leq c \varphi_{B_{2r}}^- \left( \int_{B_{2r}} |Du| dx \right) + c.$$

*The constants  $\sigma$  and  $c$  depend only on  $n, p, q, L, L_\omega, Q$  and  $[u]_{\gamma, B_{2r}}$ .*

**Proof** By the Caccioppoli estimate (3.10) and the Sobolev–Poincaré inequality (from Remark 3.7 and Lemma 3.8(1)), we obtain

$$\int_{B_\rho} \varphi(x, |Du|) dx \leq c_1 \left( \int_{B_{2\rho}} \varphi(x, |Du|)^{\frac{1}{1+\theta_1}} dx \right)^{1+\theta_1} + 1$$

for every ball  $B_{2\rho} \subset B_{2r}$ , where  $\theta_1$  and  $c_1$  depend on the parameters listed in the statement. By Gehring’s Lemma (e.g. [31, Theorem 6.6]), there exists  $\sigma > 0$  depending on  $\theta_1$  and  $c_1$  such that

$$\left( \int_{B_\rho} \varphi(x, |Du|)^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \lesssim \int_{B_{2\rho}} \varphi(x, |Du|) dx + 1$$

for every ball  $B_{2\rho} \subset B_{2r}$ . Moreover, using the technique from [41, Lemma 4.7] with  $(A1-\frac{n}{1-\gamma})$  instead of (A1) and with (3.13) in  $B_\rho$ , we obtain for every ball  $B_{2\rho} \subset B_{2r}$  that

$$\left( \int_{B_\rho} \varphi(x, |Du|)^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \lesssim \varphi_{B_{2\rho}}^+ \left( \int_{B_{2\rho}} |Du| dx \right) + 1 \lesssim \varphi_{B_{2\rho}}^- \left( \int_{B_{2\rho}} |Du| dx \right) + 1.$$

□

### 4 Growth functions and autonomous problems

Let us precisely define our solutions and minimizers. Since we only consider local versions we will drop the word “local” later on, as indicated by the parentheses. We say that  $u \in W_{loc}^{1,1}(\Omega)$  is a (local) weak solution to  $(\operatorname{div} A)$  if  $|Du| |A(\cdot, Du)| \in L_{loc}^1(\Omega)$  and

$$\int_{\Omega} A(x, Du) \cdot D\zeta dx = 0$$

for all  $\zeta \in W^{1,1}(\Omega)$  with  $\operatorname{supp} \zeta \Subset \Omega$  and  $|D\zeta| |A(\cdot, D\zeta)| \in L^1(\Omega)$ . We say that  $u \in W_{loc}^{1,1}(\Omega)$  is a (local) minimizer if  $F(\cdot, Du) \in L_{loc}^1(\Omega)$  and

$$\int_{\operatorname{supp}(u-v)} F(x, Du) dx \leq \int_{\operatorname{supp}(u-v)} F(x, Dv) dx$$

for every  $v \in W_{loc}^{1,1}(\Omega)$  with  $\operatorname{supp}(u - v) \Subset \Omega$ . Note that if  $(\operatorname{div} A)$  is an Euler–Lagrange equation, that is, if  $A = D_\xi F$  for some a function  $F$ , then the weak solution to  $(\operatorname{div} A)$  is a minimizer of  $(\min F)$ .

We introduce fundamental assumptions on  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $F : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$  with respect to the gradient variable  $\xi$  from [41, 42], so-called  $(p, q)$ -growth and quasi-isotropy conditions (parts (Aii) and (Aiii) of the definition, respectively). Here, “quasi-isotropic” indicates that  $A$  or  $F$  can be estimated by a non-autonomous isotropic  $\Phi$ -function, the so-called growth function.

**Definition 4.1** We say that  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $F : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$  has quasi-isotropic  $(p, q)$ -growth if conditions (Ai)–(Aiii) or (Fi)–(Fii) hold, respectively.

- (Ai) For every  $x \in \Omega$ ,  $A(x, 0) = 0$  and  $A(x, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$  and for every  $\xi \in \mathbb{R}^n$ ,  $A(\cdot, \xi)$  is measurable.
- (Aii) There exist  $L \geq 1$  and  $1 < p \leq q$  such that the radial function  $t \mapsto |D_\xi A(x, te)|$  satisfies (A0), (aInc) $_{p-2}$  and (aDec) $_{q-2}$  with the constant  $L$ , for every  $x \in \Omega$  and  $e \in \partial B_1(0)$ .
- (Aiii) There exists  $L \geq 1$  such that

$$|D_\xi A(x, \xi')| |\tilde{\xi}|^2 \leq L |D_\xi A(x, \xi)| \tilde{\xi} \cdot \tilde{\xi}$$

for all  $x \in \Omega$ ,  $\xi, \xi', \tilde{\xi} \in \mathbb{R}^n \setminus \{0\}$  with  $|\xi| = |\xi'|$ .

- (Fi) For every  $x \in \Omega$ ,  $F(x, 0) = |D_{\xi} F(x, 0)| = 0$  and  $F(x, \cdot) \in C^2(\mathbb{R}^n \setminus \{0\})$  and for every  $\xi \in \mathbb{R}^n$ ,  $F(\cdot, \xi)$  is measurable.
- (Fii) The derivative  $A := D_{\xi} F$  satisfies conditions (Aii) and (Aiii).

The assumptions (Aii) and (Aiii) for  $A := D_{\xi} F$  impose the two crucial conditions on the Hessian matrix  $D_{\xi}^2 F$ . The former means that  $|\xi|^2 D_{\xi}^2 F(x, \xi)$  satisfies a  $(p, q)$ -growth condition which is a variant of  $(p, q)$ -growth of  $F$ . The latter is equivalent to the existence of  $\tilde{L} \geq 1$  such that

$$\frac{\sup\{\text{eigenvalue of } D_{\xi}^2 F(x, \xi) : \xi \in \partial B_t(0)\}}{\inf\{\text{eigenvalue of } D_{\xi}^2 F(x, \xi) : \xi \in \partial B_t(0)\}} \leq \tilde{L} \quad \text{for all } x \in \Omega \text{ and } t > 0. \tag{4.2}$$

Note that all examples in Table 1 satisfy this condition, but not

$$\frac{\sup\{\text{eigenvalue of } D_{\xi}^2 F(x, \xi) : x \in \Omega, \xi \in \partial B_t(0)\}}{\inf\{\text{eigenvalue of } D_{\xi}^2 F(x, \xi) : x \in \Omega, \xi \in \partial B_t(0)\}} \leq \tilde{L} \quad \text{for all } t > 0, \tag{4.3}$$

which holds for  $F(x, \xi) \approx a(x)\psi(|\xi|)$  with  $0 < v \leq a \leq L$ . We remark (4.2) and (4.3) are called the *pointwise* and *global* uniform ellipticity condition. Here, “uniform” is concerned with the variable  $|\xi|$ . For more discussion about this uniform ellipticity condition, we refer to [22].

**Definition 4.4** Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$  have quasi-isotropic  $(p, q)$ -growth. We say that  $\varphi \in \Phi_c(\Omega)$  is its *growth function* if there exist  $1 < p_1 \leq q_1$  and  $0 < v \leq \Lambda$  such that  $\varphi(x, \cdot) \in C^1([0, \infty))$  for every  $x \in \Omega$  and  $\varphi'$  satisfies (A0), (Inc) $_{p_1-1}$  and (Dec) $_{q_1-1}$  as well as

$$|A(x, \xi)| + |\xi| |D_{\xi} A(x, \xi)| \leq \Lambda \varphi'(x, |\xi|) \quad \text{and} \quad D_{\xi} A(x, \xi) \tilde{\xi} \cdot \tilde{\xi} \geq v \frac{\varphi'(x, |\xi|)}{|\xi|} |\tilde{\xi}|^2$$

for all  $x \in \Omega$  and  $\xi, \tilde{\xi} \in \mathbb{R}^n \setminus \{0\}$ ; in the case of  $F$  we assume that the the inequalities hold for  $A := D_{\xi} F$ .

In this paper we always use the growth function from the following proposition. Thus the additional parameters  $p_1, q_1, v$  and  $\Lambda$  only depend on the original parameters  $p, q$  and  $L$ .

**Proposition 4.5** (Proposition 3.3, [42]) Every  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$  with quasi-isotropic  $(p, q)$ -growth has a growth function  $\varphi \in \Phi_c(\Omega)$  with  $p_1 = p, q_1 \geq q, v$  and  $\Lambda$  depending only on  $p, q$  and  $L$ .

By Proposition 2.3, the growth function  $\varphi$  satisfies (A0), (Inc) $_p$ , (Dec) $_{q_1}$  as well as  $\varphi^*(x, \varphi'(x, t)) \leq \varphi'(x, t)t \approx \varphi(x, t)$ . Furthermore, by [42, Remark 3.4] we have the strict monotonicity condition

$$(A(x, \xi) - A(x, \tilde{\xi})) \cdot (\xi - \tilde{\xi}) \gtrsim \frac{\varphi'(x, |\xi| + |\tilde{\xi}|)}{|\xi| + |\tilde{\xi}|} |\xi - \tilde{\xi}|^2, \quad x \in \Omega, \quad \xi, \tilde{\xi} \in \mathbb{R}^n \setminus \{0\}, \tag{4.6}$$

as well as the equivalences

$$\varphi(x, |\xi|) \approx A(x, \xi) \cdot \xi \approx |\xi| |A(x, \xi)| \quad \text{and} \quad F(x, \xi) \approx \varphi(x, |\xi|), \quad x \in \Omega, \quad \xi \in \mathbb{R}^n, \tag{4.7}$$

where the implicit constants depend on only  $p, q$  and  $L$ .

Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  have quasi-isotropic  $(p, q)$ -growth and growth function  $\varphi$  and let  $u \in W_{loc}^{1,1}(\Omega)$  be a weak solution to  $(\operatorname{div} A)$ . We showed in [42, Section 4.1] that  $u$  is a quasiminimizer of the  $\varphi$ -energy for some  $Q = Q(p, q, L) \geq 1$ ; in the reference we assumed that smooth functions are dense in the Sobolev space  $W^{1,\varphi}$ , which is reasonable if  $\varphi$  satisfies (A1). But this is not needed here due to our changed test function class in the definition of weak solution. Thus we can apply the regularity results for quasiminimizers from Sect. 3 to weak solutions.

We next show how the (A1)-type condition of  $A$  or  $F$  transfers to the growth function  $\varphi$ . Based on Part (2), we can say that (VA1- $\psi$ ) and (wVA1- $\psi$ ) are weaker in the minimization case than in the PDE case. This justifies studying minimizers separately.

**Proposition 4.8** *Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $F : \Omega \times \mathbb{R} \rightarrow [0, \infty)$  have quasi-isotropic  $(p, q)$ -growth and growth function  $\varphi$ , and let  $\psi : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$ .*

- (1)  *$A$  or  $F$  satisfies (A1- $\psi$ ) if and only if  $\varphi$  satisfies (A1- $\psi$ ).*
- (2) *If  $A := D_\xi F$  satisfies (VA1- $\psi$ ), then so does  $F$ , with the same  $\omega$  up to a constant depending on  $p, q$  and  $L$ .*
- (3) *If  $A$  or  $F$  satisfies (VA1- $s$ ),  $s > 0$ , and  $\theta \in (0, 1]$ , then*

$$|A(x, \xi) - A(y, \xi)| \leq c \omega(r)^\theta (\varphi'(y, |\xi|) + 1)$$

or

$$|F(x, \xi) - F(y, \xi)| \leq c \omega(r)^\theta (\varphi(y, |\xi|) + 1)$$

for  $B_r$  with  $r \in (0, 1]$ ,  $x, y \in B_r \cap \Omega$  and  $\xi \in \mathbb{R}^n$  with  $3^n \omega(r)^{n(1-\theta)} |\xi|^s \in [0, |B_r|^{-1}]$ . The constant  $c > 0$  depends only on  $p, q$  and  $L$ .

**Proof** We present only the proofs for  $A$ , as the ones for  $F$  are similar but simpler. All implicit constants in the proof depend only on  $p, q$  and  $L$ . Fix  $x, y \in B_r \cap \Omega$  and  $\xi \in \mathbb{R}^n$  with  $\psi(y, \xi) \in [0, |B_r|^{-1}]$ .

We first prove (1). Suppose that  $A$  satisfies (A1- $\psi$ ) and abbreviate  $t := |\xi|$ . By (A1- $\psi$ ) of  $A$  and the equivalence (4.7),

$$\begin{aligned} \varphi(x, t) &\approx t|A(x, \xi)| \leq t|A(x, \xi) - A(y, \xi)| + t|A(y, \xi)| \\ &\lesssim t(|A(y, \xi)| + 1) + \varphi(y, t) \lesssim \varphi(y, t) + t \lesssim \varphi(y, t) + 1; \end{aligned}$$

in the last inequality we used (A0) and (Inc)<sub>1</sub> of  $\varphi$  when  $t > 1$ . Thus  $\varphi$  satisfies (A1- $\psi$ ). Conversely, suppose  $\varphi$  satisfies (A1- $\psi$ ). Then we see, using also (A0) and (aInc)<sub>1</sub>, that

$$|A(x, \xi)| \approx \frac{\varphi(x, |\xi|)}{|\xi|} \lesssim \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ \frac{\varphi(y, |\xi|) + 1}{|\xi|} \lesssim \frac{\varphi(y, |\xi|)}{|\xi|} \approx |A(y, \xi)| & \text{if } |\xi| > 1. \end{cases}$$

Hence  $A$  satisfies (A1- $\psi$ ).

We omit the proof of (2) which is essentially the same as [42, Proposition 3.8].

To prove (3), we note by (VA1- $s$ ) and Proposition 3.3 that  $|A(x, \xi) - A(y, \xi)| \leq \omega(r)^\theta (|A(y, \xi)| + 1) \approx \omega(r)^\theta (\varphi'(y, |\xi|) + 1)$  when  $|\xi|$  satisfies the condition.  $\square$

When considering equations with nonlinearity  $A$  it is natural to assume (VA1- $\psi$ ) for the function  $A^{(-1)}(x, \xi) := |\xi|A(x, \xi)$ . This is the route we took in [42]. However, the next result shows that the conditions for  $A$  and  $A^{(-1)}$  are equivalent, up to an exponent which does not affect the conclusion of the main results in the next section. Thus we will in the rest of article use the assumptions directly for  $A$ .

**Proposition 4.9** *Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  have quasi-isotropic  $(p, q)$ -growth. Then  $A$  satisfies (VA1- $\psi$ ) if and only if  $A^{(-1)}$  does. If the original modulus continuity is  $\omega$ , then the new modulus continuity can be taken as  $c\omega^{\frac{1}{p'}}$  for some  $c = c(p, q, L, L_\omega) > 0$ .*

**Proof** Let  $x, y \in B_r \cap \Omega$  and  $\psi(y, \xi) \in (0, |B_r|^{-1}]$ . Assume  $A$  satisfies (VA1- $\psi$ ). Since  $|\xi| \leq c|\xi||A(y, \xi)| + 1$ ,

$$\begin{aligned} |A^{(-1)}(x, \xi) - A^{(-1)}(y, \xi)| &= |\xi||A(x, \xi) - A(y, \xi)| \\ &\leq \omega(r)(|\xi||A(y, \xi)| + |\xi|) \leq c\omega(r)\left(|A^{(-1)}(y, \xi)| + 1\right) \end{aligned}$$

This gives (VA1- $\psi$ ) of  $A^{(-1)}$ , with modulus of continuity  $c\omega$ .

We prove opposite implication through three cases. If  $|\xi| \geq 1$ , then

$$|\xi||A(x, \xi) - A(y, \xi)| \leq \omega(r)(|\xi||A(y, \xi)| + 1) \leq \omega(r)|\xi|(|A(y, \xi)| + 1).$$

Hence we suppose that  $|\xi| < 1$ . If further  $|A(x, \xi)|, |A(y, \xi)| \leq \omega(r)^{\frac{1}{p'}}$ , then

$$|A(x, \xi) - A(y, \xi)| \leq 2\omega(r)^{\frac{1}{p'}}.$$

Otherwise, we use (A0) and (aInc) $_{p-1}$  to deduce  $c|\xi|^{p-1} \geq \max\{|A(x, \xi)|, |A(y, \xi)|\} \geq \omega(r)^{\frac{1}{p'}}$ . Thus  $1 \leq c|\xi|\omega(r)^{-\frac{1}{p}}$  so that

$$|\xi||A(x, \xi) - A(y, \xi)| \leq \omega(r)(|\xi||A(y, \xi)| + 1) \leq c\omega(r)^{1-\frac{1}{p}}|\xi|(|A(y, \xi)| + 1).$$

Dividing both sides by  $|\xi|$  gives the desired estimate. □

**Remark 4.10** In [42, Proposition 3.8] we assumed that  $A^{(-1)}$  satisfies (wVA1) but in the proof we used the condition for  $A$ . This mistake can be corrected by means of the previous proposition.

Having defined the structure conditions, we first consider quasi-isotropic  $(p, q)$ -growth for an autonomous function  $\bar{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  via its trivial extension  $\bar{A}(x, \xi) := \bar{A}(\xi)$ . It has a growth function  $\bar{\varphi} \in \Phi_c \cap C^1([0, \infty))$ , cf. [42]. For such  $\bar{A}$  and  $\bar{\varphi}$ , we present regularity results of weak solutions to

$$\operatorname{div} \bar{A}(D\bar{u}) = 0 \quad \text{in } B_r. \tag{div \bar{A}}$$

Note that we use the bar-symbol to indicate the autonomous versions of  $A, F, \varphi$  and corresponding solutions or minimizers  $u$ . In the Uhlenbeck case  $\bar{A}(\xi) = \frac{\bar{\varphi}(|\xi|)}{\xi} \xi$ , we proved the next result in [41] and in [42] we sketched how to extend the proof to the quasi-isotropic case. Since the result is independent of the (A1)-type assumptions, it applies directly also to this paper.

**Lemma 4.11** ( $C^{1,\alpha}$ -regularity, Lemma 4.4, [42]) *Let  $\bar{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  have quasi-isotropic  $(p, q)$ -growth and  $\bar{\varphi} \in \Phi_c$  be its growth function. If  $\bar{u} \in W^{1,\bar{\varphi}}(B_r)$  is a weak solution to (div  $\bar{A}$ ), then  $D\bar{u} \in C^{0,\bar{\alpha}}_{\text{loc}}(B_r, \mathbb{R}^n)$  for some  $\bar{\alpha} \in (0, 1)$  with the following estimates:*

$$\sup_{B_{\rho/2}} |D\bar{u}| \leq c \int_{B_\rho} |D\bar{u}| dx \quad \text{and} \quad \int_{B_{\tau\rho}} |D\bar{u} - (D\bar{u})_{B_{\tau\rho}}| dx \leq c\tau^{\bar{\alpha}} \int_{B_\rho} |D\bar{u}| dx$$

for every  $B_\rho \subset B_r$  and  $\tau \in (0, 1)$ . Here  $\bar{\alpha}$  and  $c > 0$  depend only on  $n, p, q$  and  $L$ .

Next, we derive a harmonic approximation lemma for weak solutions and minimizers of autonomous problems. We start by recalling a Lipschitz truncation lemma, which is a formulation from [27, Theorem 3.2] and [8, Theorem 5.2] of the result in [1], see also [19, 25].

**Lemma 4.12** *For  $w \in W_0^{1,1}(B_r)$  and  $\lambda > 0$  there exist  $w_\lambda \in W_0^{1,\infty}(B_r)$  and a zero-measure set  $N$  such that  $\|Dw_\lambda\|_{L^\infty(B_r)} \leq c\lambda$  for some  $c > 0$  depending only on  $n$  and*

$$\{w_\lambda \neq w\} \subset \{Mw > \lambda\} \cup N,$$

where  $M$  is the Hardy–Littlewood maximal operator,  $Mw(x) := \sup_{\rho>0} \int_{B_\rho(x)} |w| dy$ .

Part (1) of the next lemma, on almost solutions, was considered in [27, Lemma 1.1] and [8, Lemma 5.1] in the Orlicz and double phase case, respectively. We streamline and generalize their argument and include also almost minimizers in Part (2). The more precise estimates will allow us to omit the re-scaling step in the proofs of the main theorems.

**Lemma 4.13 (Harmonic approximation)** *Let  $\bar{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $\bar{F} : \mathbb{R}^n \rightarrow [0, \infty)$  have quasi-isotropic  $(p, q)$ -growth,  $\bar{\varphi} \in \Phi_c$  be its growth function and  $\bar{u} \in u + W_0^{1,\bar{\varphi}}(B_r)$  be a weak solution to  $(\operatorname{div} \bar{A})$ , for  $\bar{A} := D_\xi \bar{F}$  in the case of  $\bar{F}$ . Suppose that there exist  $b, \sigma > 0$  and  $\delta \in (0, 1)$  for which*

$$\int_{B_r} \bar{\varphi}(|Du|)^{1+\sigma} dx \leq b^{1+\sigma}$$

and one of the following conditions holds for all  $\eta \in W_0^{1,\infty}(B_r)$ :

- (1)  $\left| \int_{B_r} \bar{A}(Du) \cdot D\eta dx \right| \leq \delta b \left\| \frac{D\eta}{\bar{\varphi}^{-1}(b)} \right\|_{L^\infty(B_r)}^\mu$  for  $\mu := 1$ .
- (2)  $\int_{B_r} \bar{F}(Du) dx \leq \int_{B_r} \bar{F}(Du + D\eta) dx + \delta b \left( \left\| \frac{D\eta}{\bar{\varphi}^{-1}(b)} \right\|_{L^\infty(B_r)} + 1 \right)^\mu$  for some  $\mu > 0$ .

Then there exist  $c = c(n, p, q, L, \sigma) > 0$  and  $\bar{\sigma} := \frac{\sigma p}{\mu + \sigma p}$  such that

$$\int_{B_r} \frac{\bar{\varphi}'(|Du| + |D\bar{u}|)}{|Du| + |D\bar{u}|} |Du - D\bar{u}|^2 dx \leq c\delta^{\bar{\sigma}} b.$$

**Proof** All implicit constants in this proof depend only on  $n, p, q, L$  and  $\sigma$ . Let  $w := u - \bar{u} \in W_0^{1,\bar{\varphi}}(B_r)$  and let  $\lambda \geq 1$  be a constant to be chosen. With these  $w$  and  $\lambda$  we consider  $w_\lambda \in W_0^{1,\infty}(B_r)$  from Lemma 4.12 so that  $\|Dw_\lambda\|_{L^\infty(B_r)} \lesssim \lambda$ . Denote

$$\mathcal{V} := \frac{\bar{\varphi}'(|Du| + |D\bar{u}|)}{|Du| + |D\bar{u}|} |Du - D\bar{u}|^2, \quad W := B_r \cap \{w \neq w_\lambda\} \quad \text{and} \quad W^c = B_r \cap \{w = w_\lambda\}.$$

Since  $\bar{\varphi}$  is independent of  $x$ , we have  $\int_{B_r} \bar{\varphi}(|D\bar{u}|)^{1+\sigma} dx \lesssim \int_{B_r} \bar{\varphi}(|Du|)^{1+\sigma} dx$  by a Calderón–Zygmund-type estimate in the Orlicz setting, see for instance [42, Lemma 4.5] with  $\theta(x, t) \equiv t^{1+\sigma}$ . This and the integrability assumption on  $Du$  imply that

$$\left( \int_{B_r} \bar{\varphi}(|D\bar{u}|) dx \right)^{1+\sigma} \leq \int_{B_r} \bar{\varphi}(|D\bar{u}|)^{1+\sigma} dx \lesssim \int_{B_r} \bar{\varphi}(|Du|)^{1+\sigma} dx \leq b^{1+\sigma}.$$

To estimate  $\frac{|W|}{|B_r|}$ , we use  $W \subset \{M(|Dw|) > \lambda\} \cup N$  from Lemma 4.12 and the maximal estimate in  $L^{\bar{\varphi}^{1+\sigma}}$  [33, Corollary 4.3.3]:

$$\begin{aligned} \frac{|W|}{|B_r|} &\leq \frac{|\{M(|Dw|) > \lambda\} \cap B_r|}{|B_r|} \leq \int_{B_r} \frac{\bar{\varphi}(M(|Dw|))^{1+\sigma}}{\bar{\varphi}(\lambda)^{1+\sigma}} dx \\ &\lesssim \frac{1}{\bar{\varphi}(\lambda)^{1+\sigma}} \int_{B_r} \bar{\varphi}(|Dw|)^{1+\sigma} dx \lesssim \frac{b^{1+\sigma}}{\bar{\varphi}(\lambda)^{1+\sigma}}, \end{aligned}$$

where, in the last step we used  $\bar{\varphi}(|Dw|) \lesssim \bar{\varphi}(|D\bar{u}|) + \bar{\varphi}(|Du|)$  and the earlier integrability estimates for  $Du$  and  $D\bar{u}$ . Using Hölder’s inequality and these estimates, we find that

$$\begin{aligned} &\int_{B_r} [\bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|) + \bar{\varphi}(\lambda)] \chi_W dx \\ &\leq \left(\frac{|W|}{|B_r|}\right)^{\frac{\sigma}{1+\sigma}} \left(\int_{B_r} (\bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|))^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} + \frac{|W|}{|B_r|} \bar{\varphi}(\lambda) \quad (4.14) \\ &\lesssim \left(\frac{b}{\bar{\varphi}(\lambda)}\right)^{\sigma} b + \frac{b^{1+\sigma}}{\bar{\varphi}(\lambda)^{1+\sigma}} \bar{\varphi}(\lambda) \approx \frac{b^{1+\sigma}}{\bar{\varphi}(\lambda)^{\sigma}}. \end{aligned}$$

We first assume (1). Using monotonicity (4.6) and that  $\bar{u}$  is a weak solution to  $(\operatorname{div} \bar{A})$ , we find that

$$\begin{aligned} \int_{B_r} \mathcal{V} \chi_{W^c} dx &\lesssim \int_{B_r} (\bar{A}(Du) - \bar{A}(D\bar{u})) \cdot (Du - D\bar{u}) \chi_{W^c} dx \\ &= \int_{B_r} \bar{A}(Du) \cdot Dw_{\lambda} dx - \int_{B_r} (\bar{A}(Du) - \bar{A}(D\bar{u})) \cdot Dw_{\lambda} \chi_W dx \end{aligned}$$

For the first term we use assumption (1) with  $\eta = w_{\lambda}$ , and for the second we use that growth functions satisfy  $|\bar{A}| \lesssim \bar{\varphi}'$ . Continuing using Young’s inequality,  $\bar{\varphi}^*(\bar{\varphi}'(t)) \lesssim \bar{\varphi}(t)$  and  $\|Dw_{\lambda}\|_{L^{\infty}(B_r)} \lesssim \lambda$ , we find that

$$\begin{aligned} \int_{B_r} \mathcal{V} \chi_{W^c} dx &\lesssim \frac{b}{\bar{\varphi}^{-1}(b)} \delta \|Dw_{\lambda}\|_{L^{\infty}(B_r)} + \int_{B_r} (\bar{\varphi}'(|Du|) + \bar{\varphi}'(|D\bar{u}|)) \|Dw_{\lambda}\|_{L^{\infty}(B_r)} \chi_W dx \\ &\lesssim \frac{b}{\bar{\varphi}^{-1}(b)} \delta \lambda + \int_{B_r} [\bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|) + \bar{\varphi}(\lambda)] \chi_W dx \end{aligned}$$

In  $W$ , we use  $\mathcal{V} \lesssim \bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|)$  and obtain the same integral as on the right-hand side, above. With these estimates and (4.14), we obtain that

$$\int_{B_r} \mathcal{V} dx \leq \int_{B_r} \mathcal{V} \chi_W dx + \int_{B_r} \mathcal{V} \chi_{W^c} dx \lesssim \frac{b}{\bar{\varphi}^{-1}(b)} \delta \lambda + \frac{b^{1+\sigma}}{\bar{\varphi}(\lambda)^{\sigma}}.$$

We choose  $\lambda := \bar{\varphi}^{-1}(b\delta^{-\kappa}) \leq \delta^{-\kappa/p} \bar{\varphi}^{-1}(b)$  with  $\kappa := \frac{p}{1+\sigma p}$ , and find that

$$\int_{B_r} \mathcal{V} dx \lesssim \left(\delta \frac{\bar{\varphi}^{-1}(b\delta^{-\kappa})}{\bar{\varphi}^{-1}(b)} + \delta^{\kappa\sigma}\right) b \lesssim (\delta^{1-\frac{\kappa}{p}} + \delta^{\kappa\sigma}) b \approx \delta^{\frac{\sigma p}{1+\sigma p}} b.$$

This is the desired upper bound with  $\bar{\sigma} := \frac{\sigma p}{1+\sigma p}$  and concludes the proof in Case (1).

Assume next that  $\bar{A} := D_{\xi} \bar{F}$  and (2) holds. We showed in [42, Lemma 6.3] that

$$\frac{\bar{\varphi}'(|\xi_1| + |\xi_2|)}{|\xi_1| + |\xi_2|} |\xi_1 - \xi_2|^2 \lesssim \bar{F}(\xi_1) - \bar{F}(\xi_2) - D_{\xi} \bar{F}(\xi_2) \cdot (\xi_1 - \xi_2).$$

Using this with  $\bar{A} = D_\xi \bar{F}$ , the weak form of  $(\operatorname{div} \bar{A})$ ,  $\bar{u} = u + w_\lambda$  in  $W^c$ ,  $\bar{F} \approx \bar{\varphi}$ , assumption (2),  $|\bar{A}| \lesssim \bar{\varphi}'$  and Young's inequality with  $\bar{\varphi}^*(\bar{\varphi}'(t)) \lesssim \bar{\varphi}(t)$ , we have

$$\begin{aligned} \int_{B_r} \mathcal{V} \chi_{W^c} \, dx &\lesssim \int_{B_r} [\bar{F}(Du) - \bar{F}(D\bar{u}) - \bar{A}(D\bar{u}) \cdot (Du - D\bar{u})] \chi_{W^c} \, dx \\ &= \int_{B_r} [\bar{F}(Du) - \bar{F}(Du + Dw_\lambda)] \, dx \\ &\quad - \int_{B_r} [\bar{F}(Du) - \bar{F}(Du + Dw_\lambda) - \bar{A}(D\bar{u}) \cdot Dw_\lambda] \chi_W \, dx \\ &\lesssim b\delta \left( \left\| \frac{Dw_\lambda}{\bar{\varphi}^{-1}(b)} \right\|_{L^\infty(B_r)} + 1 \right)^\mu + \int_{B_r} [\bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|) + \bar{\varphi}(\lambda)] \chi_W \, dx. \end{aligned}$$

In  $W$  we use the estimate  $\mathcal{V} \lesssim \bar{\varphi}(|Du|) + \bar{\varphi}(|D\bar{u}|)$  as before. With (4.14) and the choice  $\lambda := \bar{\varphi}^{-1}(b\delta^{-\kappa}) \lesssim \delta^{-\kappa/p} \bar{\varphi}^{-1}(b)$  for  $\kappa := \frac{p}{\mu + \sigma p}$ , we obtain that

$$\int_{B_r} \mathcal{V} \, dx \lesssim b\delta \left( \frac{\lambda}{\bar{\varphi}^{-1}(b)} + 1 \right)^\mu + \frac{b^{1+\sigma}}{\bar{\varphi}(\lambda)^\sigma} \lesssim (\delta^{1-\frac{\kappa\mu}{p}} + \delta^{\sigma\kappa})b \approx \delta^{\frac{\sigma p}{\mu + \sigma p}} b.$$

This is the desired upper bound with  $\bar{\sigma} := \frac{\sigma p}{\mu + \sigma p}$  and concludes the proof in Case (2).  $\square$

### 5 Maximal regularity

Let  $B_r \Subset \Omega$  with  $|B_r| \leq 1$ ,  $\gamma \in (0, 1)$  and

$$t_K := K|B_r|^{\frac{\gamma-1}{n}} \quad \text{for fixed } K \geq 1.$$

For  $\varphi \in \Phi_c(\Omega)$  with  $\varphi'$  satisfying (A0), (Inc) $_{p-1}$  and (Dec) $_{q_1-1}$  with  $1 < p \leq q_1$  we define

$$\bar{\varphi}(t) := \int_0^t \bar{\varphi}'(s) \, ds \quad \text{with} \quad \bar{\varphi}'(t) := \begin{cases} \varphi'(x_0, t) & \text{if } 0 \leq t \leq t_K, \\ \frac{\varphi'(x_0, t_K)}{t_K^{p-1}} t^{p-1} & \text{if } t_K \leq t, \end{cases} \quad (5.1)$$

where  $x_0$  is the center of  $B_r = B_r(x_0)$ . The relationship between  $\varphi$  and  $\bar{\varphi}$  is analogous to that in [41, Section 5] with  $t_1 = 0$  and  $t_2 = t_K$ . Due to our improved tools, we are able to simplify the argument concerning small values of  $t$  by removing the case  $t \in [0, t_1]$ .

**Proposition 5.2** *Let  $t_K$ ,  $\varphi$  and  $\bar{\varphi}$  be as above. Suppose that  $\varphi^+(t) \leq L_K \varphi^-(t)$  for some  $L_K \geq 0$  and all  $t \in [1, t_K]$ .*

- (1)  $\bar{\varphi} \in C^1([0, \infty))$  and  $\bar{\varphi}'$  satisfies (Inc) $_{p-1}$  and (Dec) $_{q_1-1}$ .
- (2)  $\bar{\varphi}(t) = \varphi(x_0, t)$  for all  $t \leq t_K$ .
- (3)  $\bar{\varphi}(t) \leq \frac{q_1}{p} L_K \varphi(x, t) + L$  for all  $(x, t) \in B_r \times [0, \infty)$  and  $W^{1,\varphi}(B_r) \subset W^{1,\bar{\varphi}}(B_r)$ .

**Proof** Parts (1) and (2) follow directly from the definition of  $\bar{\varphi}$  and the inclusion in (3) follows from the inequality. If  $t \leq 1$ , then the inequality in (3) follows from (A0) and when  $t \in [1, t_K]$ , it follows from  $\varphi^+(t) \leq L_K \varphi^-(t)$  and (2). If  $t > t_K$ , we calculate

$$\begin{aligned} \bar{\varphi}(t) &= \varphi(x_0, t_K) + \int_{t_K}^t \frac{\varphi'(x_0, t_K)}{t_K^{p-1}} s^{p-1} \, ds = \varphi(x_0, t_K) + \frac{t^p - t_K^p}{t_K^p} \frac{t_K \varphi'(x_0, t_K)}{p} \\ &\leq \frac{q_1}{p} \left( \frac{t}{t_K} \right)^p \varphi(x_0, t_K) \leq \frac{q_1}{p} L_K \left( \frac{t}{t_K} \right)^p \varphi_{B_r}^-(t_K) \leq \frac{q_1}{p} L_K \varphi(x, t). \quad \square \end{aligned}$$

We prove our main theorem on Hölder continuous weak solutions to  $(\operatorname{div} A)$ . The major novelties as follows: We use Proposition 3.3 to deal with large values of the derivative; this allows us to handle the borderline double phase case  $q - p = \frac{\alpha}{1-\gamma}$  with  $a \in VC^{0,\alpha}$ . Second, this is the first time that harmonic approximation from [8] has been applied in the generalized Orlicz case. Third, optimizations in the formulations allow us to avoid many steps in previous proofs and present a much more streamlined argument. Fourth, we obtain  $C^{1,\alpha}$ -regularity with exponent  $\alpha$  independent of the a priori information  $[u]_\gamma$ .

**Theorem 5.3** *Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  have quasi-isotropic  $(p, q)$ -growth and  $u \in W_{\operatorname{loc}}^{1,1}(\Omega) \cap C^{0,\gamma}(\Omega)$  be a weak solution to  $(\operatorname{div} A)$  with  $\gamma \in (0, 1)$ .*

- (1) *If  $A$  satisfies  $(VA1-\frac{n}{1-\gamma})$ , then  $u \in C_{\operatorname{loc}}^{0,\alpha}(\Omega)$  for every  $\alpha \in (0, 1)$ .*
- (2) *If  $A$  satisfies  $(VA1-\frac{n}{1-\gamma})$  with  $\omega(r) \lesssim r^\beta$  for some  $\beta > 0$ , then  $u \in C_{\operatorname{loc}}^{1,\alpha}(\Omega)$  for some  $\alpha = \alpha(n, p, q, L, \gamma, \beta) \in (0, 1)$ .*

**Proof** We prove the result in three steps. All implicit constants in the following estimates depend only on  $n, p, q, L$  and  $[u]_\gamma$ .

*Step 1, setting and approximating equation.* For  $\omega$  from  $(VA1-\frac{n}{1-\gamma})$  we fix  $B_{4r} \Subset \Omega$  with

$$r \in (0, 1), \quad \omega(4r) \leq 1 \quad \text{and} \quad \omega(r)^{1-\theta} \leq \frac{1}{3}, \tag{5.4}$$

where  $\theta \in (0, 1)$  is given in Step 2, below. Since we always consider  $B_r$  with  $\omega(4r) \leq 1$ , we assume without loss of generality that  $A$  satisfies  $(A1-\frac{n}{1-\gamma})$  with  $L_\omega = 1$ . We set

$$K := 3^{-(1-\gamma)}\omega(r)^{-(1-\theta)(1-\gamma)} \quad \text{so that} \quad t_K = 3^{-(1-\gamma)}\omega(r)^{-(1-\theta)(1-\gamma)}|B_r|^{\frac{\gamma-1}{n}}.$$

Let  $\varphi \in \Phi_c(\Omega)$  be the growth function of  $A$  from Proposition 4.5 so that  $\varphi(x, \cdot) \in C^1([0, \infty))$  and  $\varphi'$  satisfies  $(A0)$ ,  $(\operatorname{Inc})_{p-1}$  and  $(\operatorname{Dec})_{q_1-1}$ . By Proposition 4.8(1)&(3),  $\varphi$  satisfies  $(A1-\frac{n}{1-\gamma})$  with  $L_\omega$  depending only on  $p, q$  and  $L$  and

$$|A(x, \xi) - A(y, \xi)| \leq \omega(r)^\theta (\varphi'(y, |\xi|) + 1) \quad \text{when} \quad |\xi| \leq t_K.$$

By  $(A0)$  of  $\varphi'$ ,  $|A| \approx \varphi'$  and  $\omega(r) \leq 1$  we conclude that

$$\varphi(x, t) \approx t|A(x, te_1)| \leq t(|A(y, te_1)| + \varphi'(y, t) + 1) \lesssim \varphi(y, t)$$

for every  $t \in [1, t_K]$  and  $x, y \in B_r$ . Thus we can apply Proposition 5.2 with  $L_K > 0$  depending only on  $p, q$  and  $L$ .

Let  $\bar{\varphi} \in \Phi_c$  be from (5.1). By [42, Lemma 5.2] with  $t_1 = 0$  and  $t_2 = t_K$  there exists an autonomous nonlinearity  $\bar{A} \in C(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$  having quasi-isotropic  $(p, q_1)$ -growth such that  $\bar{\varphi}$  is its growth function and

$$\bar{A}(\xi) = A(x_0, \xi) \quad \text{whenever} \quad |\xi| \leq \frac{1}{2}t_K.$$

The exact form of  $\bar{A}$  is given in [42, (5.2)].

By Theorem 3.14,  $\varphi(\cdot, |Du|) \in L^{1+\sigma}(B_r)$  for some  $\sigma > 0$  depending only on  $n, p, q, L$ , and by (3.13) in  $B_{2r}$ ,

$$\int_{B_{2r}} |Du| dx \lesssim r^{\gamma-1} \lesssim t_K. \tag{5.5}$$

Let  $\bar{u} \in W^{1,\bar{\varphi}}(B_r)$  be the weak solution to  $(\operatorname{div} \bar{A})$  in  $B_r$  with boundary value  $u$ . Then  $\bar{u}$  is a quasiminimizer of the  $\bar{\varphi}$ -energy and so the results of Sect. 3 can be applied. By Jensen's

inequality, the minimization property and Proposition 5.2(3),

$$\bar{\varphi} \left( \int_{B_r} |D\bar{u}| dx \right) \leq \int_{B_r} \bar{\varphi}(|D\bar{u}|) dx \lesssim \int_{B_r} \bar{\varphi}(|Du|) dx \lesssim \int_{B_r} \varphi(x, |Du|) dx + 1.$$

Then we use the reverse Hölder inequality (Theorem 3.14), (A0), Proposition 5.2(2) with (5.5) and (Dec)<sub>q<sub>1</sub></sub> of  $\varphi$  to conclude that

$$\bar{\varphi} \left( \int_{B_r} |D\bar{u}| dx \right) \lesssim \varphi_{B_{2r}}^- \left( \int_{B_{2r}} |Du| dx + 1 \right) \approx \bar{\varphi} \left( \int_{B_{2r}} |Du| dx + 1 \right).$$

It follows that

$$\int_{B_r} |D\bar{u}| dx \lesssim \int_{B_{2r}} |Du| dx + 1 \lesssim r^{\gamma-1}. \tag{5.6}$$

Next we set

$$J := \bar{\varphi}^{-1} \left( \left( \int_{B_r} \varphi(x, |Du|)^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} + 1 \right).$$

By the reverse Hölder inequality (Theorem 3.14) and the earlier estimate (5.6),

$$J \approx \bar{\varphi}^{-1} \left( \varphi_{B_{2r}}^- \left( \int_{B_{2r}} |Du| dx \right) + 1 \right) \approx \int_{B_{2r}} |Du| dx + 1 \lesssim r^{\gamma-1}, \tag{5.7}$$

and, since  $J \gtrsim 1$ ,

$$\varphi_{B_r}^+(J) \approx \varphi_{B_r}^-(J) \approx \bar{\varphi}(J) \approx \varphi_{B_r}^- \left( \int_{B_{2r}} |Du| dx + 1 \right) \lesssim \varphi_{B_r}^-(r^{\gamma-1}). \tag{5.8}$$

*Step 2, harmonic approximation.* We prove that  $u$  is an almost weak solution to (div  $\bar{A}$ ) in the sense that

$$\left| \int_{B_r} \bar{A}(Du) \cdot D\eta dx \right| \leq c\bar{\omega}(r) \frac{\bar{\varphi}(J)}{J} \|D\eta\|_\infty \quad \text{with} \quad \bar{\omega}(r) := \omega(r)^{\frac{(1-\gamma)\sigma(p-1)}{(1-\gamma)\sigma(p-1)+1}}$$

for some  $c \geq 1$  depending on  $n, p, q, L$  and  $[u]_{\gamma,4r}$  and all  $\eta \in W_0^{1,\infty}(B_r)$ .

It suffices to consider  $\eta$  with  $\|D\eta\|_\infty \leq 1$  by scaling. Since  $u$  is a weak solution to (div  $A$ ),

$$\begin{aligned} \left| \int_{B_r} \bar{A}(Du) \cdot D\eta dx \right| &\leq \left| \int_{B_r} A(x, Du) \cdot D\eta dx \right| + \int_{B_r} |A(x, Du) - \bar{A}(Du)| dx \\ &= \int_{E_1 \cup E_2} |A(x, Du) - \bar{A}(Du)| dx, \end{aligned}$$

where  $E_1 := \{x \in B_r : |Du| \leq \frac{1}{2}t_K\}$  and  $E_2 := \{x \in B_r : |Du| > \frac{1}{2}t_K\}$ .

We first consider  $E_1$  so that  $\bar{A} = A(x_0, \cdot)$ . By Proposition 4.8(3) with  $s := \frac{n}{1-\gamma}$ ,

$$\begin{aligned} \int_{B_r} |A(x, Du) - \bar{A}(Du)|_{\chi_{E_1}} dx &= \int_{B_r} |A(x, Du) - A(x_0, Du)|_{\chi_{E_1}} dx \\ &\lesssim \omega(r)^\theta \left( \int_{B_r} \varphi'(x, |Du|) dx + 1 \right). \end{aligned}$$

We abbreviate  $\hat{\varphi} := \varphi_{B_r}^+$  and estimate the integral on the right-hand side by  $\varphi'(x, t) \approx \varphi(x, t)/t$ , Jensen's inequality for  $\hat{\varphi}^*$ ,  $\hat{\varphi}^*(\varphi(x, t)/t) \leq \varphi(x, t)$ , Hölder's inequality and estimate (5.8) for  $J$ :

$$\int_{B_r} \varphi'(x, |Du|) dx \lesssim (\hat{\varphi}^*)^{-1} \left( \int_{B_r} \varphi(x, |Du|) dx \right) \lesssim (\hat{\varphi}^*)^{-1} (\bar{\varphi}(J)) \lesssim \frac{\bar{\varphi}(J)}{J}.$$

Since  $1 \lesssim \frac{\bar{\varphi}(J)}{J}$ , the whole integral over  $E_1$  can be bounded by  $\omega(r)^\theta \frac{\bar{\varphi}(J)}{J}$ .

In  $E_2$  we estimate

$$\bar{\varphi}(\frac{1}{2}tK)^\sigma \int_{B_r} \varphi(x, |Du|) \chi_{E_2} dx \leq \int_{B_r} \varphi(x, |Du|)^{1+\sigma} dx \lesssim \bar{\varphi}(J)^{1+\sigma}.$$

Since  $\frac{\bar{\varphi}(J)}{\bar{\varphi}(\frac{1}{2}tK)} \lesssim \frac{\bar{\varphi}(r^{\gamma-1})}{\bar{\varphi}(Kr^{\gamma-1})} \leq K^{-p}$  by (5.7) and (Inc)<sub>p</sub>, we obtain

$$\int_{B_r} \varphi(x, |Du|) \chi_{E_2} dx \lesssim K^{-\sigma p} \bar{\varphi}(J). \tag{5.9}$$

By Proposition 5.2(3) and (A0),  $\bar{\varphi}(t) \lesssim \varphi(x, t)$  when  $t \geq 1$ . Using this, the  $\hat{\varphi}^*$ -Jensen inequality as in  $E_1$ , the previous estimate and (alnc)<sub>1/p'</sub> of  $(\hat{\varphi}^*)^{-1}$ , we find that

$$\begin{aligned} \int_{B_r} |A(x, Du) - \bar{A}(Du)| \chi_{E_2} dx &\lesssim \int_{B_r} \frac{\varphi(x, |Du|) + \bar{\varphi}(|Du|)}{|Du|} \chi_{E_2} dx \lesssim \int_{B_r} \frac{\varphi(x, |Du|)}{|Du|} \chi_{E_2} dx \\ &\lesssim (\hat{\varphi}^*)^{-1} \left( \int_{B_r} \varphi(x, |Du|) \chi_{E_2} dx \right) \\ &\lesssim K^{-\sigma(p-1)} (\hat{\varphi}^*)^{-1} (\bar{\varphi}(J)) \approx K^{-\sigma(p-1)} \frac{\bar{\varphi}(J)}{J}, \end{aligned}$$

where we used  $(\hat{\varphi}^*)^{-1}(\hat{\varphi}(t)) \approx \hat{\varphi}(t)/t$  and (5.8) in the last step. The desired estimate follows when we combine the estimates in  $E_1$  and  $E_2$ , recall that  $K \geq 3^{-1}\omega(r)^{-(1-\theta)(1-\gamma)}$  and choose  $\theta := \frac{(1-\gamma)\sigma(p-1)}{(1-\gamma)\sigma(p-1)+1}$ .

*Step 3, conclusion.* Applying Lemma 4.13(1) with  $b := \bar{\varphi}(J)$  to the inequality from Step 2, we obtain that

$$\int_{B_r} \frac{\bar{\varphi}(|D\bar{u}| + |Du|)}{|D\bar{u}| + |Du|} |Du - D\bar{u}|^2 dx \lesssim \bar{\omega}(r)^{\bar{\sigma}} \bar{\varphi}(J) \approx \bar{\omega}(r)^{\bar{\sigma}} \bar{\varphi} \left( \int_{B_{2r}} |Du| dx + 1 \right),$$

where  $\bar{\sigma} = \frac{\sigma p}{1+\sigma p}$ . By well-known techniques (see, e.g., [41, Corollary 6.3] for details) this implies an  $L^1$ -estimate for the difference of the gradients of  $u$  and  $\bar{u}$  from Step 1:

$$\int_{B_r} |Du - D\bar{u}| dx \leq c \bar{\omega}(r)^{\frac{\bar{\sigma}}{2q_1}} \left( \int_{B_{2r}} |Du| dx + 1 \right), \tag{5.10}$$

with  $c \geq 1$  depending on  $n, p, q, L$  and  $[u]_{\gamma, 4r}$  and  $\bar{\omega}$  from Step 2. Note that we can make  $\bar{\omega}(r)^{\frac{\bar{\sigma}}{2q_1}}$  as small as we want by choosing  $r$  small. Therefore, this inequality and the Lipschitz regularity of the  $\bar{A}$ -solutions  $\bar{u}$  (the first estimate in Lemma 4.11) with (5.6) imply local  $C^{0,\alpha}$ -regularity for every  $\alpha \in (0, 1)$  by known methods, see, e.g., [41, Theorem 7.2].

We next assume that  $\omega(r) \lesssim r^\beta$  for some  $\beta > 0$ . Fix  $\Omega' \Subset \Omega$ . From Part (1) we obtain  $u \in C^{0,\gamma'}(\Omega')$  for any  $\gamma' \in (\gamma, 1)$ , and consider  $B_{4r} \subset \Omega'$  with  $r$  satisfying (5.4) and  $[u]_{\gamma, 4r, \Omega'} \leq 1$ . Then we obtain (5.10) with  $\bar{\omega}(r)^{\frac{\bar{\sigma}}{2q_1}} \lesssim r^{\beta_0}$  for  $\beta_0$  depending only  $n, p, q, L, \gamma$  and  $\beta$ . This inequality and the Hölder regularity of the gradient of the  $\bar{A}$ -solution  $\bar{u}$  (the second estimate in Lemma 4.11) with (5.6) imply  $u \in C^{1,\alpha}_{loc}(\Omega')$  for any  $\alpha \in (0, \min\{\beta_0, 1 - \gamma'\})$

by known methods, see, e.g., [41, Theorem 7.4]. Since  $\beta_0$  and  $1 - \gamma'$  are independent of the arbitrary set  $\Omega' \Subset \Omega$ , this implies that  $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$  depending only on  $n, p, q, L, \gamma$  and  $\beta$ .  $\square$

Next we prove maximal regularity for Hölder continuous minimizers of (min  $F$ ). This is our second main result.

**Theorem 5.11** *Let  $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  have quasi-isotropic  $(p, q)$ -growth and  $u \in W_{\text{loc}}^{1,1}(\Omega) \cap C^{0,\gamma}(\Omega)$  be a minimizer of (min  $F$ ) with  $\gamma \in (0, 1)$ .*

- (1) *If  $F$  satisfies (VA1- $\frac{n}{1-\gamma}$ ), then  $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$  for every  $\alpha \in (0, 1)$ .*
- (2) *If  $F$  satisfies (VA1- $\frac{n}{1-\gamma}$ ) with  $\omega(r) \lesssim r^\beta$  for some  $\beta > 0$ , then  $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$  for some  $\alpha = \alpha(n, p, q, L, \gamma, \beta) \in (0, 1)$ .*

**Proof** The methodology is similar to Theorem 5.3 except for the application of harmonic approximation. Hence we will take advantage many parts of that proof.

*Step 1, setting and approximating functional.* We use the same choice of  $r$  and  $K$  as in Theorem 5.3 and define  $J$  in the same way. Let  $\varphi \in \Phi_c(\Omega)$  be the growth function of  $F$  from Proposition 4.5; it satisfies the same properties as in Theorem 5.3. In [42, Lemma 5.3] we constructed an autonomous function  $\bar{F} : \mathbb{R}^n \rightarrow [0, \infty)$  such that  $\bar{\varphi} \in \Phi_c$  from (5.1) is its growth function and

$$\bar{F}(\xi) = F(x_0, \xi) \quad \text{whenever } |\xi| \leq \frac{1}{2}t_K.$$

By Theorem 3.14,  $\varphi(\cdot, |Du|) \in L^{1+\sigma}(B_r)$  for some  $\sigma = \sigma(n, p, q, L, [u]_{\gamma,4r}) \in (0, 1)$ . Let  $\bar{u} \in W^{1,\bar{\varphi}}(B_r)$  be the minimizer of

$$\min_{\bar{u} \in W_0^{1,\bar{\varphi}}(B_r)} \int_{B_r} \bar{F}(D\bar{u}) \, dx, \tag{5.12}$$

or, equivalently, the weak solution to (div  $\bar{A}$ ) in  $B_r$  with  $\bar{A} := D_\xi \bar{F}$  and boundary value given by  $u$ .

*Step 2, harmonic approximation.* We prove that  $u$  is an almost minimizer of (5.12) in the sense that there exists  $c = c(n, p, q, L, [u]_{\gamma,4r}) \geq 1$  such that

$$\int_{B_r} \bar{F}(Du) \, dx \leq \int_{B_r} \bar{F}(Du + D\eta) \, dx + c\bar{\omega}(r) \left( \frac{\|D\eta\|_\infty}{J} + 1 \right)^{(1+\sigma)q_1} \bar{\varphi}(J)$$

for every  $\eta \in W_0^{1,\infty}(B_r)$ , where  $\bar{\omega}(r) := \omega(r)^{\frac{(1-\gamma)\sigma p}{(1-\gamma)\sigma p + 1}}$ .

Let  $E_1$  and  $E_2$  be the sets from Step 2 of the proof of Theorem 5.3. By  $\bar{F} \approx \bar{\varphi}$  (4.7), the definition of  $\bar{F}$ , Propositions 5.2(3) and 4.8(3), and the definition of  $J$ ,

$$\begin{aligned} \int_{B_r} \bar{F}(Du) \, dx &\leq \int_{B_r} [F(x_0, Du)\chi_{E_1} + c\varphi(x, |Du|)\chi_{E_2}] \, dx \\ &\leq \int_{B_r} F(x, Du) \, dx + c\omega(r)^\theta \int_{B_r} [\varphi(x, |Du|) + 1] \, dx + cK^{-\sigma p} \bar{\varphi}(J) \\ &\leq \int_{B_r} F(x, Du) \, dx + c(\omega(r)^\theta + K^{-\sigma p}) \bar{\varphi}(J), \end{aligned}$$

where the estimate for the term in  $E_2$  is from (5.9).

Next we obtain a similar estimate for  $v := u + \eta \in u + W_0^{1,\infty}(B_r)$ . We define sets  $E'_i$  like  $E_i$  but with  $|Du|$  replaced by  $|Dv|$ . Since  $u$  is an  $F$ -minimizer and  $F \approx \varphi$  (4.7),

$$\int_{B_r} F(x, Du) dx \leq \int_{B_r} F(x, Dv) dx \leq \int_{B_r} F(x, Dv)\chi_{E'_1} dx + c \int_{B_r} \varphi(x, |Dv|)\chi_{E'_2} dx.$$

In  $E'_1$  we use Proposition 4.8(3) with  $\bar{F} = F(x_0, \cdot)$ :

$$\int_{B_r} F(x, Dv)\chi_{E'_1} dx \leq \int_{B_r} [\bar{F}(Dv) + c\omega(r)^\theta(\varphi(x, |Dv|) + 1)] dx.$$

For the second term on the right-hand side, we use  $v = u + \eta$  and  $\varphi(x, |Dv|) \lesssim \varphi(x, |Du|) + \varphi(x, |D\eta|)$ . Thus

$$\int_{B_r} \varphi(x, |Dv|) dx \lesssim \bar{\varphi}(J) + \int_{B_r} \varphi(x, |D\eta|) dx.$$

We use (Dec) $_{q_1}$  of  $\varphi$  along with (5.8) to handle the integral with  $D\eta$ :

$$\int_{B_r} \varphi(x, |D\eta|) dx \leq \left( \int_{B_r} \varphi(x, |D\eta|)^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \lesssim \left( \frac{\|D\eta\|_\infty}{J} + 1 \right)^{q_1} \bar{\varphi}(J).$$

In  $E'_2$  we estimate

$$\begin{aligned} \bar{\varphi}(\frac{1}{2}t_K)^\sigma \int_{B_r} \varphi(x, |Dv|)\chi_{E'_2} dx &\leq \int_{B_r} \varphi(x, |Dv|)^{1+\sigma} dx \\ &\lesssim \bar{\varphi}(J)^{1+\sigma} + \int_{B_r} \varphi(x, |D\eta|)^{1+\sigma} dx. \end{aligned}$$

With the estimate for  $D\eta$  from the previous paragraph, this and  $\frac{\bar{\varphi}(J)}{\bar{\varphi}(\frac{1}{2}t_K)} \lesssim K^{-p}$  give

$$\int_{B_r} \varphi(x, |Dv|)\chi_{E'_2} dx \lesssim K^{-\sigma p} \left( \frac{\|D\eta\|_\infty}{J} + 1 \right)^{(1+\sigma)q_1} \bar{\varphi}(J).$$

Collecting the estimates from this and the previous paragraph, we arrive at

$$\int_{B_r} F(x, Du) dx \leq \int_{B_r} \bar{F}(Dv) dx + c (\omega(r)^\theta + K^{-\sigma p}) \left( \frac{\|D\eta\|_\infty}{J} + 1 \right)^{(1+\sigma)q_1} \bar{\varphi}(J).$$

Combining this with the previous paragraphs, the estimate  $K \geq 3^{-1}\omega(r)^{-(1-\theta)(1-\gamma)}$  and the choice of  $\theta := \frac{(1-\gamma)\sigma p}{(1-\gamma)\sigma p + 1}$ , we complete this step.

*Step 3, conclusion.* We apply Lemma 4.13(2) with  $b := \bar{\varphi}(J)$  and  $\mu := (1 + \sigma)q_1$  to conclude that

$$\int_{B_r} \frac{\bar{\varphi}'(|Du| + |D\bar{u}|)}{|Du| + |D\bar{u}|} |Du - D\bar{u}|^2 dx \lesssim \bar{\omega}(r)^{\bar{\sigma}} \left( \bar{\varphi} \left( \int_{B_{2r}} |Du| dx \right) + 1 \right)$$

where  $\bar{\sigma} := \frac{\sigma p}{(1+\sigma)q_1 + \sigma p}$  and  $\bar{\omega}$  is from Step 2. The estimate

$$\int_{B_r} |Du - D\bar{u}| dx \leq c\bar{\omega}(r)^{\frac{\bar{\sigma}}{2q_1}} \left( \int_{B_{2r}} |Du| dx + 1 \right)$$

follows from this as in [41, Corollary 6.3]. We complete the proof as in Step 3 of the proof of Theorem 5.3. Hence we omit details. □

We apply the previous theorem to the double phase energies from Example 3.2.

**Corollary 5.13** *Let  $1 < p \leq q$ ,  $a \geq 0$  and  $\gamma \in (0, 1)$ . Assume that either*

- $F(x, \xi) := |\xi|^p + a(x)|\xi|^q$ , where  $a \in VC^{0,\alpha}(\Omega)$ ,  $\alpha \in (0, 1)$  and  $q - p \leq \frac{1}{1-\gamma}\alpha$ ; or
- $F(x, \xi) := |\xi|^p + a(x)^\alpha |\xi|^q$ , where  $a \in C^{0,1}(\Omega)$ ,  $\alpha > 1$  and  $q - p < \frac{1}{1-\gamma}\alpha$ .

*If  $u \in W_{loc}^{1,1}(\Omega) \cap C^{0,\gamma}(\Omega)$  is a minimizer of  $(\min F)$ , then  $u \in C_{loc}^{1,\beta}(\Omega)$  for some  $\beta$  depending only on  $n, p, q, \alpha$  and  $\gamma$ .*

**Proof** Consider first the case with  $\alpha < 1$ . By Example 3.2,  $F$  satisfies (VA1- $\frac{n}{1-\gamma}$ ). Therefore  $u \in C_{loc}^{0,\beta}(\Omega)$  for every  $\beta \in (0, 1)$  by the previous theorem. Fix  $\gamma_1 \in (\gamma, 1)$ . Then  $F$  satisfies (VA1- $\frac{n}{1-\gamma_1}$ ) with  $\omega(r) \lesssim r^{\alpha-(q-p)(1-\gamma_1)}$  and  $\alpha - (q-p)(1-\gamma_1) > \alpha - (q-p)(1-\gamma) \geq 0$ . Since  $u \in C_{loc}^{0,\gamma_1}(\Omega)$ , the previous theorem yields that  $u \in C_{loc}^{1,\beta}(\Omega)$  for some  $\beta \in (0, 1)$  depending on  $n, p, q, L, \alpha$  and  $\gamma$ . In the case  $\alpha > 1$  we can directly apply the theorem since the strict inequality  $q - p < \frac{1}{1-\gamma}\alpha$  implies that the condition (VA1- $\frac{n}{1-\gamma}$ ) holds with a  $\omega$  of power-type (cf. Example 3.2). □

Finally, combining Theorems 3.12, 5.3 and 5.11, we obtain regularity results for BMO and  $L^{s^*}$  weak solutions and minimizers. Note that the case  $L^{s^*}$  with  $s \leq (n(1 - \frac{p}{q}))$  remains open, due to a lack of a Sobolev–Poincaré inequality.

**Corollary 5.14** *Let  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  or  $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  have quasi-isotropic  $(p, q)$ -growth and satisfy (wVA1- $s$ ). Suppose that  $u \in W_{loc}^{1,1}(\Omega)$  is a weak solution to  $(\operatorname{div} A)$  or a minimizer of  $(\min F)$  and that one of the following holds:*

- (1)  $s = n$  and  $u \in BMO(\Omega)$ ,
- (2)  $s \in (n(1 - \frac{p}{q}), n)$  and  $u \in L^{s^*}(\Omega)$ .

*Then  $u \in C_{loc}^{0,\alpha}(\Omega)$  for every  $\alpha \in (0, 1)$ .*

*Furthermore, if (VA1- $s'$ ) holds with  $\omega(r) \lesssim r^\beta$  for some  $s' > s$  and  $\beta > 0$ , then  $u \in C_{loc}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$  depending on  $n, p, q, L, s, s'$  and  $\beta$ .*

**Proof** Note that (wVA1- $s$ ) implies (A1- $s$ ) and fix  $\Omega' \Subset \Omega$ . Since  $u$  is a quasiminimizer of the isotropic problem, Theorem 3.12 implies that  $u \in C^{0,\gamma}(\Omega')$  for some  $\gamma \in (0, 1)$ . The condition (wVA1- $s$ ) with  $s < \frac{n}{1-\gamma}$  implies (VA1- $\frac{n}{1-\gamma}$ ). By Theorem 5.3 or 5.11, we obtain  $u \in C_{loc}^{0,\alpha}(\Omega')$  so that  $u \in C_{loc}^{0,\alpha}(\Omega)$  for every  $\alpha \in (0, 1)$ .

Next, we suppose that  $\omega(r) \lesssim r^\beta$  in (VA1- $s'$ ) and fix  $\Omega' \Subset \Omega$ . By the previous part,  $u \in C^{0,\gamma}(\Omega')$  for any  $\gamma \in (0, 1)$ . Furthermore, (VA1- $\frac{n}{1-\gamma}$ ) holds with  $\omega(r) \lesssim r^\beta$  when  $\gamma$  is chosen so large that  $\frac{n}{1-\gamma} \geq s'$ . Therefore, by Theorem 5.3 or 5.11, we obtain  $u \in C_{loc}^{1,\alpha}(\Omega')$  so that  $u \in C_{loc}^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$  depending only on  $n, p, q, L, \gamma$  and  $\beta$ . □

**Remark 5.15** For  $C^{1,\alpha}$ -regularity, the obtained exponent  $\alpha$  is independent of the a priori information on weak solutions or minimizers.

**Acknowledgements** Peter Hästö was supported in part by the Jenny and Antti Wihuri Foundation and Jihoon Ok was supported by the National Research Foundation of Korea by the Korean Government (NRF-2022R1C1C1004523). We thank the referee for comments.

**Funding** Open Access funding provided by University of Turku (including Turku University Central Hospital).

**Data availability** Data sharing is not applicable to this article as obviously no datasets were generated or analyzed during the current study.

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