



On the surjunctivity and the Garden of Eden theorem for non-uniform cellular automata

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Abstract

Non-uniform cellular automata (NUCA) are an extension of cellular automata with multiple local rules in different cells. We show that if the distribution of local rules is uniformly recurrent, or recurrent in the one-dimensional case, the Garden of Eden theorem holds. We also show that for any one-dimensional non-recurrent distribution, there is a substitution of local rules that defines a NUCA which does not satisfy the Garden of Eden theorem. Finally, we show that a one-dimensional rule distribution asymptotic to recurrent distribution defines a surjunctive NUCA.

Keywords Non-uniform cellular automata · Garden of Eden theorem · Surjunctivity

1 Introduction

The Garden of Eden theorem states that a cellular automaton, or CA for short, is surjective if and only if it is pre-injective. One direction of the Garden of Eden theorem for CA over integer grids was first proved by Moore in 1962 and the other direction by Myhill in 1963. It is known that there are groups that don't satisfy the Garden of Eden theorem, specifically, the theorem is satisfied exactly when the group is amenable (Ceccherini-Silberstein and Coornaert 2010; Moore 1962; Myhill 1963).

Surjunctivity is a broader notion that was introduced by Gottschalk in 1973. A group is said to be surjunctive if all CA over the group have the property that they are surjective if they are injective. It is known that all sofic groups are surjunctive, but it remains an open problem whether all groups are surjunctive (Gottschalk 1973; Ceccherini-Silberstein and Coornaert 2010).

Finally, non-uniform cellular automata, or NUCA, are a generalization of regular CA which operate with different local rules in different cells. The local rules are given by a

special configuration called a local rule distribution. It can easily be seen that the Garden of Eden theorem does not hold in general for non-uniform CA over integer grids. More recently, conditions for the surjunctivity of non-uniform CA have been researched (Sipper 1996; Dennunzio et al. 2012; Phung 2023).

In this work we show that if a NUCA is defined by a uniformly recurrent local distribution, it satisfies the Garden of Eden theorem. More specifically in the 1-dimensional case, we show that the theorem is satisfied if the distribution is recurrent. Conversely, we show that for any one-dimensional non-recurrent configuration, there is an assignment of local rules to symbols such that the obtained rule distribution defines a NUCA which does not satisfy the Moore direction of the Garden of Eden theorem. We also prove the analogous result for the Myhill direction. Finally we show that all distributions asymptotic to recurrent distributions are surjunctive, that is, they define NUCA which are surjective if they are injective.

2 General definitions

We define some general notation regarding configurations.

Definition 1 Let Σ be a finite set called a *state set* and its elements *states*. Let $d \in \mathbb{Z}_+$. A *configuration* is a function

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$c : \mathbb{Z}^d \rightarrow \Sigma$ and d is the *dimension* of the configuration. An element $\bar{x} \in \mathbb{Z}^d$ is called a *cell* and $c(\bar{x})$ is the *state of cell* \bar{x} .

The space $\Sigma^{\mathbb{Z}^d}$ come equipped with Cantor’s topology, and is known to be compact.

Definition 2 Let Σ be a state-set and $d \in \mathbb{Z}_+$. A finite set $D \subseteq \mathbb{Z}^d$ is called a *finite domain*. A *finite pattern* of domain D is a function $p : D \rightarrow \Sigma$.

Let $c \in \Sigma^{\mathbb{Z}^d}$. The pattern $c|_D \in \Sigma^D$ is the unique pattern for which it holds that $c|_D(\bar{x}) = c(\bar{x})$ for all $\bar{x} \in D$.

Definition 3 Let $D \subseteq \mathbb{Z}^d$ be a finite domain, $p_1 \in \Sigma^D$ and $\bar{r} \in \mathbb{Z}^d$. Let $E = \{\bar{x} + \bar{r} \mid \bar{x} \in D\}$ and $p_2 \in \Sigma^E$. The pattern p_2 is a *translated copy* of p_1 if $p_1(\bar{x}) = p_2(\bar{x} + \bar{r})$ for all $\bar{x} \in D$.

Definition 4 For any $x, y \in \mathbb{Z}$, we denote $[x, y] = \{z \in \mathbb{Z} \mid x \leq z \leq y\}$. A *hypercube* $C \subseteq \mathbb{Z}^d$ is a finite domain such that for some $w \in \mathbb{Z}_+$ and cell $\bar{x} \in \mathbb{Z}^d$,

$$C = \{\bar{x} + (v_1, \dots, v_d) \mid \forall i \in [1, d] : 0 \leq v_i < w\}.$$

The constant w is called the *width* of C , and C is called *w-wide*. Recurrence and uniform recurrence are general properties of a dynamical system. The definitions here are equivalent to the usual definitions for configurations under a shift dynamic.

Definition 5 A configuration $c \in \Sigma^{\mathbb{Z}^d}$ is *recurrent* if for all finite domains $D \subseteq \mathbb{Z}^d$, there is $E \subseteq \mathbb{Z}^d$ such that $D \neq E$ and $c|_E$ is a translated copy of $c|_D$. In other words, a configuration is recurrent if every finite pattern that appears in the configuration, appears at least twice (and hence, appears infinitely many times).

The configuration c is *uniformly recurrent* if for every finite domain $D \subseteq \mathbb{Z}^d$, there exists $w \in \mathbb{Z}_+$ such that for any hypercube of width w , the hypercube contains a translated copy of $c|_D$. In other words, a configuration is uniformly recurrent if every pattern that appears in it, appears in every hypercube of width w .

3 Non-uniform cellular automata

Non-uniform cellular automata are a generalization of cellular automata with the possibility of having multiple different local rules.

Definition 6 Let Σ be a state set, $d \in \mathbb{Z}_+$ a dimension and $N = (\bar{n}_1, \dots, \bar{n}_m)$ a tuple of vectors $\bar{n}_i \in \mathbb{Z}^d$, $1 \leq i \leq m$ called a *neighbourhood*. A *local rule* is a function $f : \Sigma^m \rightarrow \Sigma$.

Notation 1 Let $N = (\bar{n}_1, \bar{n}_2, \dots, \bar{n}_m)$ be a neighbourhood, $\bar{x} \in \mathbb{Z}^d$ and $D \subseteq \mathbb{Z}^d$. The neighbourhood of a cell is denoted as

$$N(\bar{x}) = \{\bar{x} + \bar{n} \mid \bar{n} \in N\}$$

and the neighbourhood of D is denoted as

$$N(D) = \{\bar{x} + \bar{n} \mid \bar{x} \in D, \bar{n} \in N\}.$$

Definition 7 (Dennunzio et al. 2013) Let N be a neighbourhood of $m \in \mathbb{Z}_+$ cells and \mathcal{R} be a finite set of local rules $\Sigma^m \rightarrow \Sigma$. A configuration $\theta \in \mathcal{R}^{\mathbb{Z}^d}$ is called a *local rule distribution*. A *non-uniform cellular automaton* or *NUCA* for short is the tuple $A = (\Sigma, d, N, \mathcal{R}, \theta)$. Let $A = (\Sigma, d, N, \mathcal{R}, \theta)$ be a NUCA where $N = (\bar{n}_1, \dots, \bar{n}_m)$.

The *global update rule* of A is the function $H_\theta : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ that maps any configuration $c \in \Sigma^{\mathbb{Z}^d}$ to the configuration $H_\theta(c)$ such that

$$H_\theta(c)(\bar{x}) = \theta(\bar{x})(c(\bar{x} + \bar{n}_1), \dots, c(\bar{x} + \bar{n}_m))$$

for all $\bar{x} \in \mathbb{Z}^d$.

If the NUCA is clear from context, it is usually referred to by its global update rule H_θ . The global update rule of a NUCA is continuous under Cantor’s topology (Dennunzio et al. 2012).

Definition 8 Let $A = (\Sigma, d, N, \mathcal{R}, \theta)$ be a NUCA where $N = (\bar{n}_1, \dots, \bar{n}_m)$. Let $D \subseteq \mathbb{Z}^d$ be a finite domain. An *update rule over domain* D is the function $H_{\theta|D} : \Sigma^{|N(D)|} \rightarrow \Sigma^D$ that maps any finite pattern $p \in \Sigma^{N(D)}$ to the pattern

$$H_{\theta|D}(p)(\bar{x}) = \theta(\bar{x})(p(\bar{x} + \bar{n}_1), \dots, p(\bar{x} + \bar{n}_m))$$

for all $\bar{x} \in D$.

We introduce the concept of local rule templates to characterize rule distributions. A local rule distribution template is a configuration of symbols which function as templates for local rules. Each symbol can then be assigned a local rule, which yields a local rule distribution. This allows for making general statements about distributions which are obtained from different assignments to the same template.

Definition 9 Let \mathcal{R} be a finite set of local rules, $d \in \mathbb{Z}_+$ a dimension and T a finite set whose elements are called *rule*

templates. A configuration $\tau \in T^{\mathbb{Z}^d}$ is called a *local rule distribution template* and a function $\alpha : T \rightarrow \mathcal{R}$ is called an *assignment of local rules*. The rule distribution $\tau_\alpha \in \mathcal{R}^{\mathbb{Z}^d}$ such that $\tau_\alpha(x) = \alpha(\tau(x))$ is called τ with assignment α .

A uniform rule distribution is a valid assignment of any rule template. A rule template is a limit on the non-uniformity of a distribution.

4 The Garden of Eden theorem

The Garden of Eden theorem, or GoE theorem for short, states that the global rule of a CA is surjective if and only if it is injective on finite configurations. This can be stated more generally by replacing the notion of injectivity on finite configurations with pre-injectivity.

Definition 10 Let $c, e \in \Sigma^{\mathbb{Z}^d}$. The *difference set* of c and e is the set

$$\text{diff}(c, e) = \{\bar{x} \in \mathbb{Z}^d \mid c(\bar{x}) \neq e(\bar{x})\}.$$

If $\text{diff}(c, e)$ is finite, c and e are called *asymptotic*.

Definition 11 Let H_θ be the update rule of a NUCA. H_θ is *pre-injective* if for all asymptotic configurations $c, e \in \Sigma^{\mathbb{Z}^d}$ such that $c \neq e$, it holds that $H_\theta(c) \neq H_\theta(e)$.

The GoE theorem is known to hold for regular CA (Moore 1962; Myhill 1963), but it is also easy to see that neither direction of the theorem holds for NUCA generally. In this section we show that the theorem holds for NUCA with uniformly recurrent distributions, and that in the 1-dimensional case, it holds if the distribution is recurrent. First we state an auxiliary lemma used in both proofs. It is a simple generalization of a lemma in Dennunzio et al. (2013).

Lemma 1 Let $\theta \in \mathcal{R}^{\mathbb{Z}^d}$. The d -dimensional global update rule H_θ is surjective if and only if the partial update rule $H_{\theta|C}$ is surjective for all hypercubes $C \subseteq \mathbb{Z}^d$

Proof Let N be the neighbourhood of H_θ . First assume H_θ is surjective. Let $C \subseteq \mathbb{Z}^d$ be a hypercube and $p \in \Sigma^C$. Clearly there is a configuration $c \in \Sigma^{\mathbb{Z}^d}$ such that $c|_C = p$. Because H_θ is surjective, there is a configuration $e \in \Sigma^{\mathbb{Z}^d}$ such that

$H_\theta(e) = c$. Then $H_{\theta|C}(e|_{N(C)}) = c|_C = p$ and therefore $H_{\theta|C}$ is surjective.

Assume then that $H_{\theta|C}$ is surjective for all hypercubes $C \subseteq \mathbb{Z}^d$. Let C_i be a width i hypercube centered on the origin for all $i \in \mathbb{Z}_+$. Let $c \in \Sigma^{\mathbb{Z}^d}$ and $(e_i)_{i=1}^\infty$ a sequence of configurations $e_i \in \Sigma^{\mathbb{Z}^d}$ such that $H_{\theta|C_i}(e_i|_{N(C_i)}) = c|_{C_i}$ for all $i \in \mathbb{Z}_+$. Because $H_{\theta|C_i}$ is surjective, such a sequence exists.

Let $c_i = H_\theta(e_i)$. Clearly the sequence $(c_i)_{i=1}^\infty$ converges with limit $\lim_{i \rightarrow \infty} c_i = c$. By compactness of the Cantor topology, the sequence $(e_i)_{i=1}^\infty$ has a converging subsequence $(e_{i_j})_{j=1}^\infty$. Let $e = \lim_{j \rightarrow \infty} e_{i_j}$. Then by the continuity of NUCA,

$$c = \lim_{i \rightarrow \infty} c_i = \lim_{j \rightarrow \infty} c_{i_j} = \lim_{j \rightarrow \infty} H_\theta(e_{i_j}) = H_\theta(e).$$

Therefore H_θ is surjective. □

Additionally, we define the support of a configuration.

Definition 12 Let $q \in \Sigma$ be a state. The q -support of a configuration $c \in \Sigma^{\mathbb{Z}^d}$ is the set

$$\text{supp}_q(c) = \{\bar{x} \in \mathbb{Z}^d \mid c(\bar{x}) \neq q\}.$$

4.1 Uniformly recurrent distributions

All NUCA with uniformly recurrent distributions satisfy the GoE theorem. The proof is a modification of the original proof for regular CA found in Moore (1962); Myhill (1963).

Lemma 2 Let $d, s, n, r \in \mathbb{Z}_+$. For all sufficiently large $k \in \mathbb{Z}_+$, it holds that

$$(s^{n^d} - 1)^{k^d} < s^{(kn-2r)^d}.$$

Proof A proof for this lemma is also found in Moore (1962). If $s = 1$, the inequality obviously holds. Assume $s > 1$. We have

$$\begin{aligned} & (s^{n^d} - 1)^{k^d} < s^{(kn-2r)^d} \\ \Leftrightarrow & \log_s (s^{n^d} - 1)^{k^d} < \log_s s^{(kn-2r)^d} \\ \Leftrightarrow & k^d \log_s (s^{n^d} - 1) < (kn - 2r)^d \\ \Leftrightarrow & \log_s (s^{n^d} - 1) < \left(\frac{kn - 2r}{k}\right)^d = \left(n - \frac{2r}{k}\right)^d. \end{aligned}$$

Because $\log_s(s^{n^d} - 1) < \log_s s^{n^d} = n^d$, there is $\varepsilon \in \mathbb{R}_+$, such that $\log_s(s^{n^d} - 1) = (n - \varepsilon)^d$. Then if $k > \frac{2r}{\varepsilon}$, we have

$$\log_s(s^{n^d} - 1) = (n - \varepsilon)^d < \left(n - \frac{2r}{k}\right)^d.$$

□

Lemma 3 *Let $\theta \in \mathcal{R}^{\mathbb{Z}^d}$ be a uniformly recurrent rule distribution. If H_θ is not surjective, then it is not pre-injective.*

Proof Let $r \in \mathbb{Z}_+$ be large enough that H_θ can be defined with radius- r local rules. Let $s = |\Sigma|$. Suppose that H_θ is not surjective. Then by Lemma 1 there's a domain $D \subseteq \mathbb{Z}^d$ such that $H_{\theta|D}$ is not surjective. Because θ is uniformly recurrent, there is $n \in \mathbb{Z}_+$ such that any n -wide hypercube in θ contains a translated copy of $\theta|D$.

Let $q \in \Sigma$ and $k \in \mathbb{Z}_+$. Let $C \subseteq \mathbb{Z}^d$ be a hypercube of width kn and C' a hypercube of width $kn - 2r$ centred on C . Let

$$K = \{c \in \Sigma^{\mathbb{Z}^d} \mid \text{supp}_q(c) \subseteq C'\}.$$

Now $|K| = s^{|C'|} = s^{(kn-2r)^d}$.

Hypercube C can be partitioned into k^d hypercubes of width n , each of which must contain a copy of $\theta|D$ in θ . Consider then the set $H_\theta(K)$. Because $H_{\theta|D}$ is not surjective, there is a finite pattern $p \in \Sigma^D$ with no $H_{\theta|D}$ pre-image. Then for each of the k^d hypercubes in C , there is at least one finite pattern with no pre-image. Because they are identical outside of C , there are at most $(s^{n^d} - 1)^{k^d}$ configurations in $H_\theta(K)$. Now by Lemma 2,

$$|H_\theta(K)| \leq (s^{n^d} - 1)^{k^d} < s^{(kn-2r)^d} = |K|$$

for sufficiently large k . Therefore there must be configurations $c_1, c_2 \in K$ such that $c_1 \neq c_2$ and $H_\theta(c_1) = H_\theta(c_2)$. Hence H_θ is not pre-injective. □

Definition 13 Let $\bar{r} \in \mathbb{Z}^d$. The \bar{r} -shift $\sigma_{\bar{r}} : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ is the function mapping $c \in \Sigma^{\mathbb{Z}^d}$ such that for all $\bar{x} \in \mathbb{Z}^d$,

$$\sigma_{\bar{r}}(c)(\bar{x}) = c(\bar{x} - \bar{r}).$$

Lemma 4 *Let $\theta \in \mathcal{R}^{\mathbb{Z}^d}$ be a uniformly recurrent rule distribution. If H_θ is not pre-injective, then it is not surjective.*

Proof Let $r \in 2\mathbb{Z}_+$ be large enough that H_θ can be defined with radius- $\frac{r}{2}$ local rules. Let $s = |\Sigma|$. Suppose that H_θ is not pre-injective. Let $c_1, c_2 \in \Sigma^{\mathbb{Z}^d}$ be asymptotic configurations such that $c_1 \neq c_2$ and $H_\theta(c_1) = H_\theta(c_2)$.

Because c_1 and c_2 are asymptotic, there is a hypercube $D \subseteq \mathbb{Z}^d$ such that $\text{diff}(c_1, c_2)$ is contained within a hypercube centered on D whose width is at least $2r$ less than the width of D . Let $n \in \mathbb{Z}_+$ be large enough that every hypercube of width n in θ must contain a translated copy of $\theta|D$.

Consider any hypercube $E \subseteq \mathbb{Z}^d$ of width n . There is $\bar{y} \in \mathbb{Z}^d$ such that $D' = \{\bar{x} + \bar{y} \mid \bar{x} \in D\} \subseteq E$ and $\theta|_{D'}$ is a copy of $\theta|D$. Let $p_1 = \sigma_{\bar{y}}(c_1)|_D$ and $p_2 = \sigma_{\bar{y}}(c_2)|_D$. Let then $e_1, e_2 \in \Sigma^{\mathbb{Z}^d}$ be configurations such that $e_{1|D} = p_1$, $e_{2|D} = p_2$ and $e_1(\bar{x}) = e_2(\bar{x})$ for any $\bar{x} \in \mathbb{Z}^d \setminus D$. Consider any cell $\bar{x} \in \mathbb{Z}^d$. If $N(\bar{x}) \cap \text{diff}(e_1, e_2) = \emptyset$, then clearly $H_\theta(e_1)(\bar{x}) = H_\theta(e_2)(\bar{x})$. If $N(\bar{x}) \cap \text{diff}(e_1, e_2) \neq \emptyset$ then \bar{x} is within $\frac{r}{2}$ cells from a differing cell, and

$$H_\theta(e_1)(\bar{x}) = H_\theta(c_1)(\bar{x} - \bar{y}) = H_\theta(c_2)(\bar{x} - \bar{y}) = H_\theta(e_2)(\bar{x}).$$

Therefore $H_\theta(e_1) = H_\theta(e_2)$, meaning that for any hypercube of width n , there are two patterns p_1 and p_2 which can be replaced with each other without affecting the image.

Let C be a hypercube of width kn for some $k \in \mathbb{Z}_+$, and C' a hypercube of width $kn - 2r$ centred on C . If H_θ is surjective, then $H_{\theta|C'}$ is surjective, meaning every pattern in the domain C' has a pre-image in the domain C . There are $s^{(kn-2r)^d}$ possible patterns in the domain C' . The hypercube C can be partitioned into k^d hypercubes of width n , each of which must contain two patterns p_1 and p_2 such that p_1 can be replaced by p_2 without affecting the image of C . If for each of these hypercubes we fix one of these two patterns to be p_1 , then every pattern in the domain C' has a pre-image with no p_2 pattern in any hypercube. There are at most $(s^{n^d} - 1)^{k^d}$ such patterns.

By Lemma 2, for sufficiently large k , it holds that $(s^{n^d} - 1)^{k^d} < s^{(kn-2r)^d}$ and therefore some pattern in the domain C' has no pre-image. Therefore $H_{\theta|C'}$ is not surjective and hence H_θ is not surjective. □

Theorem 5 *Let $\theta \in \mathcal{R}^{\mathbb{Z}^d}$ be a uniformly recurrent rule distribution. The NUCA H_θ is surjective if and only if it is pre-injective.*

Fig. 1 The hypercube C defined in the proofs of Lemma 3 and Lemma 4, in dimension 2. Each of the smaller hypercubes it is partitioned into contains a copy of $\theta|_D$



Proof The statement follows from Lemmas 3 and 4. \square

4.2 The 1-dimensional case

In the 1-dimensional case, the distribution needs only be recurrent for the GoE theorem to hold. The proof is again a modification of the original proof for regular CA.

Lemma 6 Let $s, n, r \in \mathbb{Z}_+$. For all sufficiently large $m \in \mathbb{Z}_+$, it holds that

$$(s^n - 1)^m s^{k-nm} < s^{k-2r},$$

for all $k \in \mathbb{Z}_+$.

Proof By Lemma 2, for all sufficiently large $m \in \mathbb{Z}_+$ it holds that

$$(s^n - 1)^m < s^{mn-2r}.$$

Then for all $k \in \mathbb{Z}_+$

$$(s^n - 1)^m s^{k-nm} < s^{mn-2r} s^{k-nm} = s^{k-2r} \quad \square$$

Lemma 7 Let $\theta \in \mathcal{R}^{\mathbb{Z}}$ be a recurrent rule distribution. If H_θ is not surjective, then it is not pre-injective.

Proof Let $r \in \mathbb{Z}_+$ be large enough that H_θ can be defined with radius- r local rules. Let $s = |\Sigma|$ be the size of the state set. Suppose that H_θ is not surjective. Then by Lemma 1 there are $i, j \in \mathbb{Z}$ such that $H_{\theta|_{[i,j]}}$ is not surjective. Let $n = |j - i| + 1$.

Let $m \in \mathbb{Z}_+$ be large enough that $(s^n - 1)^m s^{k-nm} < s^{k-2r}$ for all $k \in \mathbb{Z}_+$. By Lemma 6, such a number m exists. Because θ is recurrent, it has infinitely many copies of the pattern $\theta|_{[i,j]}$. Let $C \subseteq \mathbb{Z}^d$ be a segment such that $\theta|_C$ contains m disjoint copies of $\theta|_{[i,j]}$. Let k be the length of C , and let C' be a length $k - 2r$ segment centered on C . Finally, let $q \in \Sigma$ and

$$K = \{c \in \Sigma^{\mathbb{Z}} \mid \text{supp}_q(c) \subseteq C'\}.$$

Now $|K| = s^{|C'|} = s^{k-2r}$. Clearly any pair of configurations from K is asymptotic. Consider the image $H_\theta(K)$. Because $H_\theta|_{[i,j]}$ is not surjective, there is a finite pattern $p \in \Sigma^{[i,j]}$ with no $H_\theta|_{[i,j]}$ pre-image. Then no translated copy can appear in the same position as a copy of $\theta|_{[i,j]}$ in any configuration in $H_\theta(K)$. Then there are at most $(s^n - 1)^m s^{k-nm}$ configurations in $H_\theta(K)$.

Therefore, due to the selection of m

$$|H_\theta(K)| \leq (s^n - 1)^m s^{k-nm} < s^{k-2r} = |K|.$$

Then there must be configurations $c_1, c_2 \in K$ such that $c_1 \neq c_2$ and $H_\theta(c_1) = H_\theta(c_2)$. Hence H_θ is not pre-injective. \square

Lemma 8 *Let $\theta \in \mathcal{R}^{\mathbb{Z}}$ be a recurrent rule distribution. If H_θ is not pre-injective, then it is not surjective.*

Proof Let $r \in 2\mathbb{Z}_+$ be large enough that H_θ can be defined with radius- $\frac{r}{2}$ local rules. Let $s = |\Sigma|$. Suppose that H_θ is not pre-injective. Let $c_1, c_2 \in \Sigma^{\mathbb{Z}}$ be asymptotic configurations such that $c_1 \neq c_2$ and $H_\theta(c_1) = H_\theta(c_2)$.

Because c_1 and c_2 are asymptotic, there is a segment $[i + r, j - r] \subseteq \mathbb{Z}$ such that $\text{diff}(c_1, c_2) \subseteq [i + r, j - r]$. Let $n = |i - j| + 1$. Let $p_1 = c_1|_{[i,j]}$ and $p_2 = c_2|_{[i,j]}$.

Let $e_1 \in \Sigma^{\mathbb{Z}}$ be a configuration containing a copy of p_1 in some segment $[i + y, j + y]$ where $y \in \mathbb{Z}$ is such that $\theta|_{[i+y,j+y]}$ is a copy of $\theta|_{[i,j]}$. Let $e_2 \in \Sigma^{\mathbb{Z}}$ be the configuration obtained by replacing the translated copy of p_1 in e_1 with a translated copy of p_2 . Consider cell $x \in \mathbb{Z}$. If $N(x) \cap \text{diff}(e_1, e_2) = \emptyset$, then clearly $H_\theta(e_1)(x) = H_\theta(e_2)(x)$. If $N(x) \cap \text{diff}(e_1, e_2) \neq \emptyset$ then the distance of x from a differing cell is at most $\frac{r}{2}$. Then $x \in [i + y + \frac{r}{2}, j + y - \frac{r}{2}]$, and

$$H_\theta(e_1)(x) = H_\theta(c_1)(x - y) = H_\theta(c_2)(x - y) = H_\theta(e_2)(x).$$

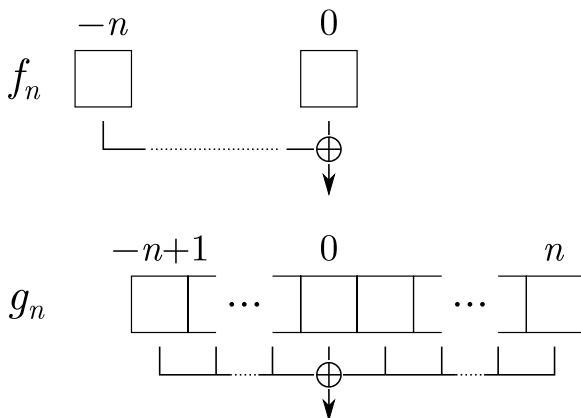


Fig. 2 The operation of rules f_n and g_n , top to bottom respectively

Therefore $H_\theta(e_1) = H_\theta(e_2)$, meaning a copy of p_1 can be replaced with a copy of p_2 without affecting the image of the configurations, provided that the copy of p_1 lies in the same cells as a copy of $\theta|_{[i,j]}$ in θ .

Let m be as in Lemma 6. Let C be a segment of such that θ_C contains m disjoint copies of $\theta|_{[i,j]}$. Since θ is recurrent, such a segment exists. Let $k \in \mathbb{Z}_+$ be the length of C . Let C' be a segment of length $kn - 2r$ centered on C . If H_θ is surjective, then $H_\theta|_{C'}$ is surjective, meaning every pattern in the domain C' has a pre-image in the domain C .

There are s^{k-2r} possible patterns in the domain C' . On the other hand, because each copy of pattern p_1 that shares its domain with one of the m disjoint copies of $\theta|_{[i,j]}$ can be replaced with a copy of p_2 without affecting the image, each pattern in C' has a pre-image with no copies of p_1 on a copy of these m disjoint positions. There are $(s^n - 1)^m s^{n(k-m)}$ such patterns. Because $(s^n - 1)^m s^{k-nm} < s^{k-2r}$, there is some pattern in C' with no pre-image. Therefore $H_\theta|_{C'}$ is not surjective and hence H_θ is not surjective. \square

Theorem 9 *Let $\theta \in \mathcal{R}^{\mathbb{Z}}$ be a recurrent rule distribution. The NUCA H_θ is surjective if and only if it is pre-injective.*

Proof The statement follows from Lemmas 7 and 8. \square

5 Non-recurrent rule distributions

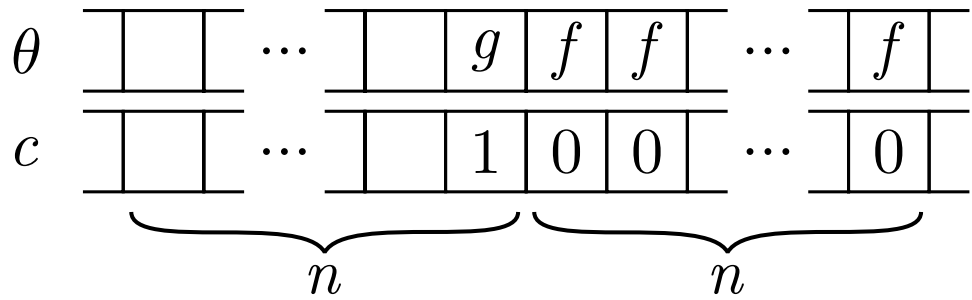
We now know that a 1-dimensional NUCA given by a recurrent rule distribution must satisfy the GoE theorem. Then, given a recurrent distribution template, any assignment to that template satisfies the theorem. In this section we show that these are in fact exactly the templates that always satisfy the theorem; for any non-recurrent template, there exists an assignment which is pre-injective, but not surjective, and an assignment which is surjective, but not pre-injective.

We begin by examining the special case of a distribution of two local rule templates, where one of the two templates can appear n times in a row at most once. We show that for such a template, for any $n \geq 1$, and for both directions of the GoE theorem, there is an assignment that does not satisfy that direction of the theorem. Then, we reduce the general case of a non-recurrent template to this special case.

5.1 Moore's theorem

In this section we present a construction for assignments to the aforementioned special case templates which give distributions whose associated global update rules are pre-injective, but not surjective. First we define some notation and terminology used in the following proofs.

Fig. 3 The rule distribution and configuration used in the argument for non-surjectivity of H_θ . Pictured is the range seen by the noted cells



Notation 2 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2$. The local rules f_n and g_n are ones with neighbourhood $N = (-n, \dots, n)$ such that

$$f_n(a_{-n}, \dots, a_n) = a_{-n} \oplus a_0,$$

$$g_n(a_{-n}, \dots, a_n) = a_{-n+1} \oplus \dots \oplus a_n.$$

An illustration of these rules is in Fig. 2.

Let $\theta \in \{f_n, g_n\}^{\mathbb{Z}}$. For each cell $x \in \mathbb{Z}$, if $\theta(x) = f_n$ we call x an f -cell and if $\theta(x) = g_n$ we call it a g -cell. We say that cell x sees cell y if $y \in N(x)$ and the state of y affects the computation of the local rule at cell x . In these terms, an f -cell x sees cells $x - n$ and x , and a g -cell sees all cells in the range $x - n + 1, \dots, x + n$.

For the sake of clarity, we omit the states of dummy neighbours, and denote

$$f(a_{-n}, a_0) = f_n(a_{-n}, \dots, a_n),$$

$$g(a_{-n+1}, \dots, a_n) = g_n(a_{-n}, \dots, a_n),$$

when n is fixed and clear from context.

Lemma 10 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2$. Let $\mathcal{R} = \{f_n, g_n\}$ and $\theta \in \mathcal{R}^{\mathbb{Z}}$ be a rule distribution with pattern $g_n(f_n)^n$. Then there is a configuration $c \in \Sigma^{\mathbb{Z}}$ with no H_θ -pre-image.

Proof Assume the n -long block of f -cells is in the segment $[x + 1, x + n]$, meaning $\theta(x) = g_n$. Let $c \in \Sigma^{\mathbb{Z}}$ be a configuration with $c(x) = 1$ and $c(y) = 0$ for all $y \in [x + 1, x + n]$. This is illustrated in Fig. 3.

Now suppose there is $e \in \Sigma^{\mathbb{Z}}$ such that $H_\theta(e) = c$. Because $c(x) = 1$, x sees an odd number of cells with state 1 in e . Therefore there are an odd number of cells with state 1 in e in the range $[x - n + 1, x + n]$.

Then consider the f -cells. Each $y \in [x + 1, x + n]$ sees $y - n$ and y . Then the f -cells in total see the cells in the ranges $[x - n + 1, x - n]$ and $[x + 1, x + n]$, covering the full range $[x - n + 1, x + n]$ seen by x . Additionally, no two cells in $[x + 1, x + n]$ see the same cell. Then because $c(y) = 0$ for all $y \in [x + 1, x + n]$, each y sees an even number of cells with state 1. This then implies there are an even

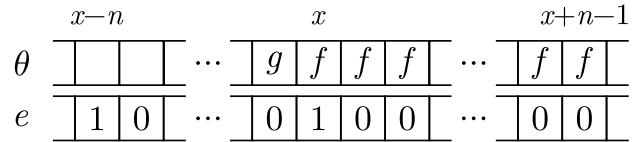


Fig. 4 The obtained θ and e . Indices of the cells are annotated on the top row

number of cells with state 1 in the range $[x - n + 1, x - n]$, which is a contradiction. Hence c has no pre-image. \square

Lemma 11 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2$. Let $\mathcal{R} = \{f_n, g_n\}$ and $\theta \in \mathcal{R}^{\mathbb{Z}}$ be such that there is at most one length n block of adjacent cells with rule f_n (and hence all other blocks of adjacent rules f_n are at most $n - 1$ cells long). Then H_θ is pre-injective.

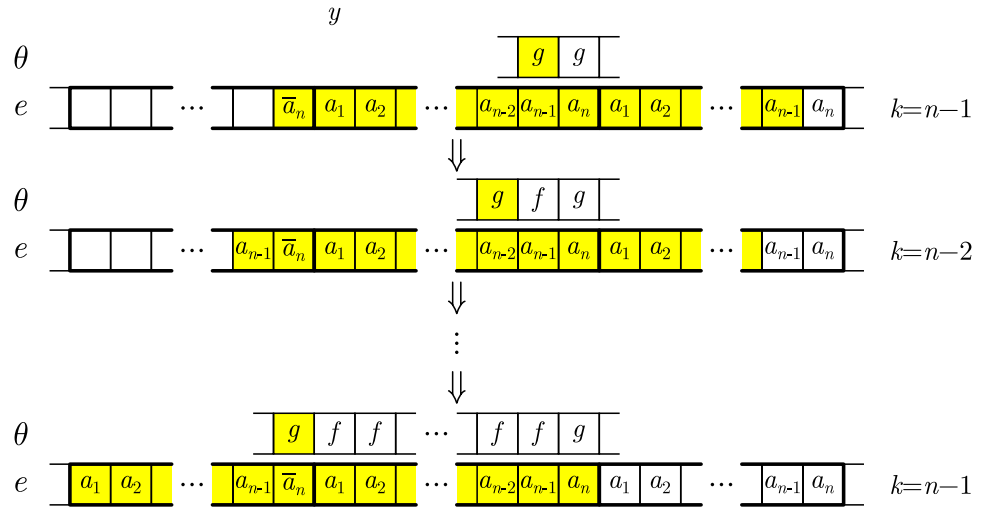
Proof Suppose $c_1, c_2 \in \Sigma^{\mathbb{Z}}$ are asymptotic and $c_1 \neq c_2$. Let $e \in \Sigma^{\mathbb{Z}}$ be the configuration with $e(x) = c_1(x) \oplus c_2(x)$. Then $e(x) = 1$ if and only if $c_1(x) \neq c_2(x)$. Assume that $H_\theta(c_1) = H_\theta(c_2)$.

Notice that $H_\theta(c_1)(x) = H_\theta(c_2)(x)$ if and only if the number of cells y seen by x such that $c_1(y) \neq c_2(y)$ is even. This is if and only if x sees an even number of cells y where $e(y) = 1$, which in turn is if and only if $H_\theta(e)(x) = 0$. Hence $H_\theta(c_1) = H_\theta(c_2)$ if and only if $H_\theta(e)(x) = 0$ for all $x \in \mathbb{Z}$.

Let $x \in \mathbb{Z}$ be the rightmost cell such that $e(x) = 1$. Such a cell exists, because c_1 and c_2 are asymptotic, but not equal. Let us prove by induction for $y = x + n - 1$ down to $y = x$ that $\theta(y) = f_n$. If $\theta(x + n - 1) = g_n$, then $H_\theta(e)(x + n - 1) = g(1, 0, \dots, 0) = 1$, which is a contradiction. Therefore $\theta(x + n - 1) = f_n$.

Suppose for some $y \in [x + 1, x + n - 1]$ and all $z \in [y, x + n - 1]$ it holds that $\theta(z) = f_n$. Then also $e(z - n) = 0$ for each z in the range, because if not, $H_\theta(e)(z) = e(z) \oplus e(z - n) = 0 \oplus 1 = 1$. Now $\theta(y - 1) = f_n$; if $\theta(y - 1) = g_n$, then $y - 1$ would see exactly one cell with state 1, meaning $H_\theta(e)(y - 1) = 1$. Then by induction, for all $y \in [x, x + n - 1]$ it holds that $\theta(y) = f_n$. Also then it holds that $e(x - n) = 1$ and $e(x - n + 1) = \dots = e(x - 1) = 0$.

Fig. 5 Illustration of the argument for why two repeated n -long patterns force a new repetition to their left. The position of cell y is annotated on the top row. The highlighted cells are seen by the highlighted rule g_n



Note that there now is a length n block of f -cells in the range $[x, x + n - 1]$, meaning that to the left of cell x there can only be at most length $n - 1$ blocks of f -cells. The obtained θ and e are illustrated in Fig. 4.

Suppose then that somewhere in e there are two copies of the same n -long segment back to back. That is, somewhere in e there appears the pattern $a_1 a_2 \dots a_n a_1 a_2 \dots a_n$. Let $y \in \mathbb{Z}$ be the leftmost cell of this pattern. Suppose also that the n -block of f -cells is somewhere to the right of cell $y + n - 1$. We show that then $e(y - 1) = a_n$. Suppose $e(y - 1) = \bar{a}_n \neq a_n$. Now $\theta(y + n - 1) = g_n$; if $\theta(y + n - 1) = f_n$ then $H_\theta(e)(y + n - 1) = f(\bar{a}_n, a_n) = 1$, a contradiction.

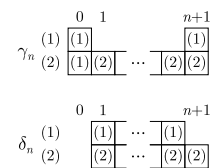
Next we show that for all $z \in [y - 1, y + n - 2]$, $\theta(z) = f_n$. Let $k \in [0, n - 1]$, and assume that for all $z \in [y + k, y + n - 2]$ it holds that $\theta(z) = f_n$. Then also it holds that for all such z , that $e(z) = e(z - n)$. Therefore $\theta(y + k - 1) = f_n$, indeed if $\theta(y + k - 1) = g_n$, then

$$\begin{aligned} H_\theta(e)(y + k - 1) &= g(a_{k+1}, \dots, a_{n-1}, \bar{a}_n, a_1, \dots, a_{n-1}, a_n, a_1, \dots, a_k) \\ &= a_1 \oplus \dots \oplus a_{n-1} \oplus \bar{a}_n \oplus a_1 \oplus \dots \oplus a_{n-1} \oplus a_n \\ &= (a_1 \oplus a_1) \oplus \dots \oplus (a_{n-1} \oplus a_{n-1}) \oplus (\bar{a}_n \oplus a_n) \\ &= 0 \oplus \dots \oplus 0 \oplus \dots \oplus 0 \oplus 1 = 1. \end{aligned}$$

Now by induction, for any $n \geq 2$ and all $z \in [y - 1, y + n - 1]$, $\theta(z) = f_n$. But this is a contradiction, because now we have a length n block of rules f_n to the left of cell $y + n - 1$ and hence $e(y - 1) = a_n$. This argument is illustrated in Fig. 5.

Next, we inductively apply the previous argument to show that there is a copy of the pattern $a_1 \dots a_n$ immediately to the left of the original pair of identical patterns. Suppose we know that immediately left of the cell y is the pattern $a_i \dots a_n$, where $i \in [2, n]$. Then we have the pattern $a_i \dots a_n a_1 \dots a_{i-1} a_i \dots a_n a_1 \dots a_{i-1}$ in e and the block of f -cells still to the right of its center, meaning by the previous,

Fig. 6 Operation of rules γ_n and δ_n , top to bottom respectively. The cells marked (1) are added together on the first track and the ones marked (2) on the second track



there is a cell with state a_{i-1} immediately to the left of this pattern. Then by induction, the statement follows.

Now by the preceding, there must be infinitely many copies of $a_1 \dots a_n$ to the left in e . Then by the first part of the proof, there are infinitely many copies of the pattern $1(0^{n-1})$ in e . This is a contradiction, because it would imply there are infinitely many cells z where $c_1(z) \neq c_2(z)$, but c_1 and c_2 were assumed to be asymptotic. Hence $H_\theta(c_1) \neq H_\theta(c_2)$ and H_θ is pre-injective. \square

5.2 Myhill's theorem

In this section we present a construction for assignments to the special case templates which give distributions whose associated global update rules are surjective, but not pre-injective. First we define some notation used in the following proofs.

Notation 3 For $a, b \in \mathbb{Z}_2$, we denote $a \oplus b = a + b \pmod 2$. For $a_1, \dots, a_n \in \mathbb{Z}_2$, we denote

$$\bigoplus_{i=1}^n a_i = a_1 \oplus \dots \oplus a_n.$$

Notation 4 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$. For any $a = (a_1, a_2) \in \Sigma$, we denote $a^{(1)} = a_1$ and $a^{(2)} = a_2$.

Definition 14 The local rules γ_n and δ_n are ones with neighbourhood $(0, \dots, n + 1)$ such that

$$\begin{aligned} \gamma_n(a_0, \dots, a_{n+1})^{(1)} &= a_0^{(1)} \oplus a_{n+1}^{(1)} \oplus a_0^{(2)}, \\ \delta_n(a_0, \dots, a_{n+1})^{(1)} &= \bigoplus_{i=1}^n a_i, \\ \gamma_n(a_0, \dots, a_{n+1})^{(2)} &= \delta_n(a_0, \dots, a_{n+1})^{(2)} = \bigoplus_{i=1}^{n+1} a_i, \end{aligned}$$

for all $a_0, \dots, a_{n+1} \in \Sigma$. These rules are illustrated in Fig. 6. Like previously, if for $x \in \mathbb{Z}$ it holds that $\theta(x) = \gamma_n$, we call x a γ -cell and if $\theta(x) = \delta_n$, we call x a δ -cell.

Lemma 12 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\mathcal{R} = \{\gamma_n, \delta_n\}$ and $\theta \in \mathcal{R}^{\mathbb{Z}}$ a rule distribution with pattern $\delta_n(\gamma_n)^n \delta_n$. Then there are asymptotic configurations $c, e \in \Sigma^{\mathbb{Z}}$ such that $c \neq e$ and $H_\theta(c) = H_\theta(e)$.

Proof Assume the length n block of γ -cells is in segment $[x, x + n - 1]$. Let $c, e \in \Sigma^{\mathbb{Z}}$ be such that $c(y) = (0, 0)$ for all $y \in \mathbb{Z}$ and $e(x + n) = (1, 0)$, $e(y') = (0, 0)$ for all $y' \in \mathbb{Z}$, $y' \neq x + n$. Clearly c and e are asymptotic and $c \neq e$, and because the local rules in θ only add modulo 2, it holds that $H_\theta(c) = c$.

Next we show that $H_\theta(e) = c$. For any cell $y \in \mathbb{Z}$ that doesn't see cell $x + n$, clearly $H_\theta(e)(y) = (0, 0)$. Consider then cells that see $x + n$. Because both γ - and δ -cells only see $n + 1$ cells to their right and none to their left, the only cells that can see $x + n$ are in the segment $[x - 1, x + n]$. For any $y \in [x, x + n - 1]$,

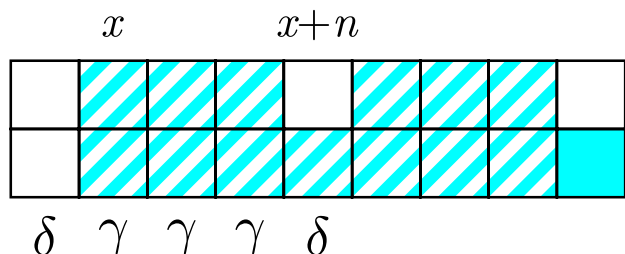


Fig. 7 The cells seen by cells in the pattern $\delta_n(\gamma_n)^n \delta_n$ in the case where $n = 3$. Cells seen by only γ -cells are highlighted in solid colour and cells seen by both δ - and γ -cells are highlighted in striped colour

$$\begin{aligned} H_\theta(e)(y)^{(1)} &= e(y)^{(1)} \oplus e(y + n)^{(1)} \oplus e(y)^{(2)} \\ &= 0 \oplus 0 \oplus 0 = 0, \\ H_\theta(e)(y)^{(2)} &= \bigoplus_{i=1}^{n+1} e(y + i)^{(2)} = 0 \oplus \dots \oplus 0 = 0, \end{aligned}$$

Because none of the cells sees the first track component of $x + n$. For cells $x - 1$ and $x + n$, we have

$$\begin{aligned} H_\theta(e)(x - 1)^{(1)} &= \bigoplus_{i=1}^n e(x - 1 + i)^{(1)} = 0 \oplus \dots \oplus 0 = 0, \\ H_\theta(e)(x + n)^{(1)} &= \bigoplus_{i=1}^n e(x + n + i)^{(1)} = 0 \oplus \dots \oplus 0 = 0, \\ H_\theta(e)(x - 1)^{(2)} &= \bigoplus_{i=1}^{n+1} e(x - 1 + i)^{(2)} = 0 \oplus \dots \oplus 0 = 0, \\ H_\theta(e)(x + n)^{(2)} &= \bigoplus_{i=1}^{n+1} e(x + n + i)^{(2)} = 0 \oplus \dots \oplus 0 = 0, \end{aligned}$$

because neither sees the first track component of $x + n$ either. Hence for all $y \in [x - 1, x + n]$, $H_\theta(y) = (0, 0)$. Therefore $H_\theta(e) = c = H_\theta(c)$. This argument is illustrated in Fig. 7. □

Lemma 13 Let $n \geq 1$ and $\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\mathcal{R} = \{\gamma_n, \delta_n\}$ and $\theta \in \mathcal{R}^{\mathbb{Z}}$ a rule distribution with at most one length n block of adjacent γ -cells. The global update rule H_θ is surjective.

Proof Let $c \in \Sigma^{\mathbb{Z}}$. We show that there is $e \in \Sigma^{\mathbb{Z}}$ such that $H_\theta(e) = c$. Our local rules have the property that flipping the state of any one cell flips the state of every cell in the image that can see it. We use this fact to construct the pre-image e cell by cell. First we fix some initial cells to state 0. We then find cells x (on either track i) in an order where each cell sees exactly one cell $e(y)^{(j)}$ where the pre-image has not been defined yet. Suppose the modulo 2 sum of the already defined cells seen by $e(x)^{(i)}$ is a . We then "reserve" $e(y)^{(j)}$ for $c(x)^{(i)}$, by defining $e(y)^{(j)} = a \oplus c(x)^{(i)}$. Then obviously $H_\theta(e)(y)^{(j)} = a \oplus e(y)^{(j)} = c(x)^{(i)}$. We proceed by induction, and thus show that a pre-image can be defined for an arbitrary configuration c .

Let $k \leq n$ be the length of the longest block of γ -cells in θ . If there are no γ -cells, it is easy to see that H_θ is surjective, so assume $k \geq 1$. For the sake of clarity we assume a block of γ -cells is in the segment $[1, k]$. This does not effect the result because the defined configuration can be shifted to match the distribution. Then let

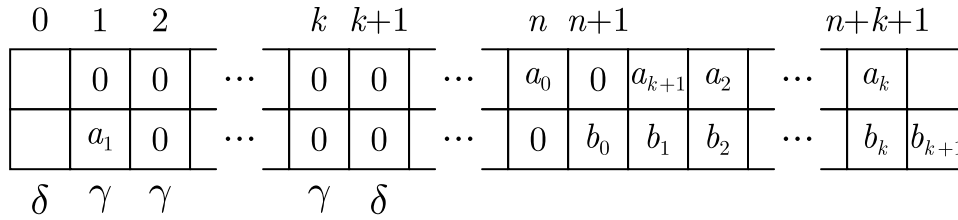


Fig. 8 Beginning the construction of the pre-image when $k < n$. The indices of cells are annotated above. Cells marked a_i and b_i are cells reserved for fixing the state of $H_\theta(e)(i)$ on track 1 and 2 respectively, where $i \in [0, k + 1]$. The rules assigned to each cell are annotated on the bottom

Fig. 9 Beginning the construction of the pre-image in the case that $k = n$

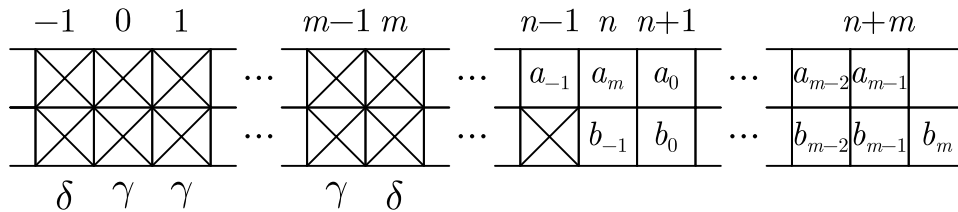
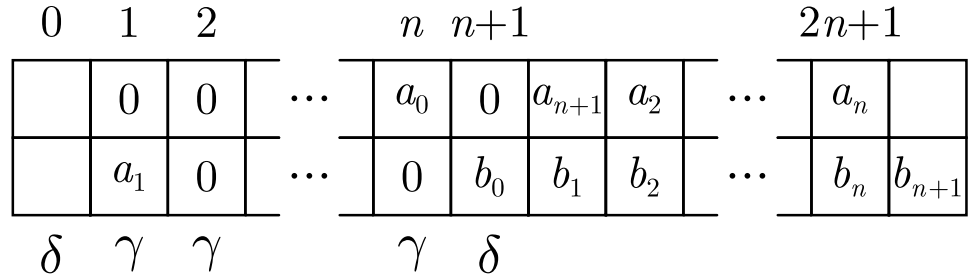


Fig. 10 Construction of the pre-image going to the right. The relative indices of cells are annotated above. Cells marked a_i and b_i are cells reserved for fixing the state of $H_\theta(e)(i)$ on track 1 and 2 respectively. Crossed out cells are cells whose state has already been defined in e

$$e(1)^{(1)} = e(2)^{(1)} = \dots = e(n - 1)^{(1)} = e(n + 1)^{(1)} \\ = e(2)^{(2)} = \dots = e(n)^{(2)} = 0,$$

and

$$e(n)^{(1)} = c(0)^{(1)}, \\ e(n + i + 1)^{(1)} = c(i)^{(1)}, \text{ when } i \in \{2, \dots, k - 1\}, \\ e(n + k + 1)^{(1)} = e(k)^{(1)} \oplus c(k)^{(1)} \\ e(n + 2)^{(1)} = \begin{cases} c(k + 1)^{(1)} \oplus (\bigoplus_{j=n+3}^{n+k+1} e(j)^{(1)}), & \text{if } k \leq n - 1 \\ c(k + 1)^{(1)} \oplus e(n)^{(1)} \oplus (\bigoplus_{j=n+3}^{n+k+1} e(j)^{(1)}), & \text{otherwise,} \end{cases} \\ e(1)^{(2)} = e(n + 2)^{(1)} \oplus c(1)^{(1)}, \\ e(n + i + 1)^{(2)} = \left(\bigoplus_{j=i+1}^{n+i} e(j)^{(2)} \right) \oplus c(i)^{(2)}, \text{ when } i \in \{0, \dots, k + 1\}.$$

Now $H_\theta(e)(x) = c(x)$ when $x \in [0, k + 1]$. For each cell $x^{(i)}$ there's a reserved fixing cell whose state is the sum of $c(x)^{(i)}$ and the states of all the other cells seen by $x^{(i)}$, modulo 2. Then because $H_\theta(e)(x)^{(i)}$ is the modulo 2 sum of the cells seen by $c(x)^{(i)}$, clearly $H_\theta(e)(x)^{(i)} = c(x)^{(i)}$. \square These states are also well defined, because their definitions only reference already defined states. For example, the track 1 cell 0 sees the cells $0, \dots, n$ on track 1, and cells $0, \dots, n - 1$ were fixed to be state 0, so we reserve the cell

n in the pre-image for fixing the state of $c(0)^{(1)}$. This construction is illustrated in Figs. 8 and 9. Next we inductively define the rest of configuration e .

Case 1 Suppose $x > k + 1$. Assume that $e(x)^{(2)}, e(x + 1)^{(2)}, \dots, e(x + n)^{(2)}$ have already been defined. Then let

$$e(x + n + 1)^{(2)} = \left(\bigoplus_{j=1}^n e(x + j)^{(2)} \right) \oplus c(x)^{(2)},$$

meaning $H_\theta(e)(x)^{(2)} = c(x)^{(2)}$. Now $e(x)^{(2)}$ is inductively defined for all $x > k + 1$.

We define the cells on track 1 by blocks of γ -cells. Assume first that $e(x)^{(1)}, e(x + 1)^{(1)}, \dots, e(x + n - 1)^{(1)}$ have already been defined. If $\theta(x) = \delta_n$ and $\theta(x - 1) = \delta_n$, let

$$e(x + n)^{(1)} = \left(\bigoplus_{j=1}^{n-1} e(x + j)^{(1)} \right) \oplus c(x)^{(1)},$$

meaning $H_\theta(e)(x)^{(1)} = c(x)^{(1)}$. If $\theta(x) = \gamma_n$ and $\theta(x - 1) = \delta_n$, then x is the leftmost cell in a length m block of γ -cells, where $1 \leq m \leq k$. Then let

$$e(x+i+n+1)^{(1)} = e(x+i)^{(1)} \oplus e(x+i)^{(2)} \oplus c(x+i)^{(1)}$$

when $i \in \{0, \dots, m-1\}$,

$$e(x+n)^{(1)} = \left(\bigoplus_{j=m+1}^{n-1} e(x+j)^{(1)} \right) \oplus \left(\bigoplus_{j=n+1}^{n+m} e(x+j)^{(1)} \right) \oplus c(x+m)^{(1)}.$$

This is illustrated in Fig. 10. These states are indeed well defined. The states $e(x), \dots, e(x+m-1)$ are defined by assumption. Using this, states $e(x+n+1), \dots, e(x+n+m)$ can be defined inductively. Then every cell seen by $x+m$ except the first track component of $x+n$ has already been defined, because the furthest right cell $x+m$ can see the first track component of is $x+n+m$.

Now $H_\theta(e)(x+i)^{(1)} = c(x+i)^{(1)}$ for all $i \in \{0, \dots, m\}$. Then by induction $H_\theta(e)(x) = c(x)$ for all $x > k+1$.

Case 2 Suppose $x < 0$. First assume that $e(x+2)^{(2)}, e(x+3)^{(2)}, \dots, e(x+n+1)^{(2)}$ have already been defined. Let

$$e(x+1)^{(2)} = c(x)^{(2)} \oplus \left(\bigoplus_{j=2}^{n+1} e(x+j)^{(2)} \right),$$

meaning $H_\theta(e)(x)^{(2)} = c(x)^{(2)}$. Now $e(x)^{(2)}$ is inductively defined for all $x < 0$.

Again, we define the cells on track 1 by blocks of γ -cells. Assume then that $e(x+2)^{(1)}, e(x+3)^{(1)}, \dots, e(x+n+1)^{(1)}$ have been defined. If $\theta(x) = \delta_n$ and $\theta(x+1) = \delta_n$, let

$$e(x+1)^{(1)} = c(x)^{(1)} \oplus \left(\bigoplus_{j=2}^n e(x+j)^{(1)} \right),$$

meaning $H_\theta(e)(x)^{(1)} = c(x)^{(1)}$. If $\theta(x) = \gamma_n$ and $\theta(x+1) = \delta_n$, then x is the rightmost cell in a length m block of γ -cells, where $1 \leq m \leq k$. Then let

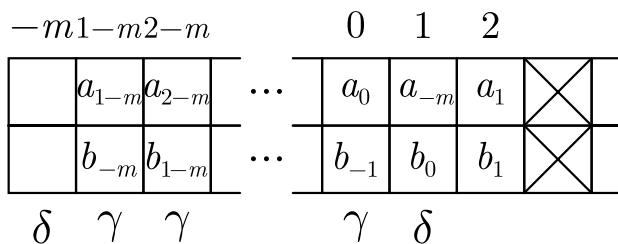


Fig. 11 Construction of the pre-image going to the left

$$e(x-i)^{(1)} = c(x-i)^{(1)} \oplus e(x-i+n+1)^{(1)} \oplus e(x-i)^{(2)}$$

when $i \in \{0, \dots, m-1\}$,

$$e(x+1)^{(1)} = \left(\bigoplus_{j=0}^{m-1} e(x-i)^{(1)} \right) \oplus \left(\bigoplus_{j=2}^{n-m} e(x+i)^{(1)} \right) \oplus c(x-m)^{(1)}.$$

This is illustrated in Fig. 11. These states are well defined: for each $i \in \{0, \dots, m-1\}$, $e(x-i)^{(2)}$ was defined earlier and $e(x-i+n+1)^{(1)}$ is defined by assumption since $m < n$. Then every cell seen by $x-m$ has been defined except for the first track component of $x+1$.

Now $H_\theta(e)(x-i)^{(1)} = c(x-i)^{(1)}$ for all $i \in \{0, \dots, m\}$. Then by induction $H(e)(x) = c(x)$ for all $x < 0$.

Now $H_\theta(e)(x) = c(x)$ for all $x \in \mathbb{Z}$, because for each $x^{(i)}$, there is a reserved fixing cell, whose state is the modulo 2 sum of $c(x)^{(i)}$ and all other states seen by $x^{(i)}$. Hence every $c \in \Sigma^{\mathbb{Z}}$ has a pre-image, meaning H_θ is surjective.

5.3 General non-recurrent templates

Next, we reduce the general case of a non-recurrent rule distribution template to the special case from Sects. 5.1 and 5.2.

Definition 15 Let T be a set of rule templates and $\tau \in T^{\mathbb{Z}}$ be a non-recurrent rule distribution template. Let $n \geq 1$ be such that there is a pattern $t_1 \dots t_n$ in τ that appears only once. Let f and g be local rules with state set Σ and neighbourhood $N = (-r, \dots, r)$ for some $r \in \mathbb{Z}_+, r \geq n$.

The rule set $\mathcal{R}_{\tau, f, g}$ is such that for each $t \in T$ there is $h_t \in \mathcal{R}_{\tau, f, g}$ with state set $\mathbb{Z}_n \times \Sigma$ and neighbourhood N which maps

$$((m_{-r}, a_{-r}), \dots, (m_r, a_r)) \mapsto (m_0, f(a_{-r}, \dots, a_r))$$

if $t = t_{m_0}$ and if in segment (m_{-n}, \dots, m_n) there is a length $n+2$ sub-segment $(a, 1, \dots, n, b)$, where $a \neq n$ and $b \neq 1$. Otherwise h_t maps

$$((m_{-r}, a_{-r}), \dots, (m_r, a_r)) \mapsto (m_0, g(a_{-r}, \dots, a_r)),$$

for all $(m_i, a_i) \in \mathbb{Z}_n \times \Sigma$.

Let's first clarify the function of the rule set $\mathcal{R}_{\tau, f, g}$. Configurations in $(\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ consist of two tracks, a static *background track*, and an *action track* that the local rules actually operate on. In the background track, the configuration contains a guess as to where in the word $t_1 \dots t_n$ the template in its cell is. Each rule in our assignment knows which template it is assigned to, so it can check if the guessed template is correct. If it is not, the rule g is used on the action track.

Fig. 12 Example of the chosen rule set and assignment, where $n = 4$. Which rules are used on the binary track are annotated on the bottom row. The string of guesses $1 \dots 4$ is highlighted. There are two correct guesses in this string and hence the rule f is used in two cells

τ	t_3	t_4	t_4	t_3	t_2	t_1	t_4	t_2	
c	2	4	3	1	2	3	4	3	
	0	0	1	0	1	1	0	1	
	g	g	g	g	f	g	f	g	

If the guess is correct, the rule further checks if the guess m is part of the sub-string $1 \dots n$ of a string $a1 \dots nb$ on the background track, where $a \neq n$ and $b \neq 1$. If it is, the rule f is used on the action track, and otherwise rule g is used. Note that the local rule does not know whether the other guesses in this string are correct, only the guess in its own cell. An example is illustrated in Fig. 12.

Notation 5 Let $\Sigma = A \times B$ for some state sets A and B . For any $c \in \Sigma^{\mathbb{Z}}$, $c^{(1)} \in A^{\mathbb{Z}}$, $c^{(2)} \in B^{\mathbb{Z}}$ are such that $c(x) = (c^{(1)}(x), c^{(2)}(x))$ for all $x \in \mathbb{Z}$.

Lemma 14 Let $\tau \in T^{\mathbb{Z}}$ be a non-recurrent rule distribution template and $n \geq 1$ such that there is a length n pattern $t_1 \dots t_n$ in τ that appears only once. Let be $\mathcal{R}_1 = \{f, g\}$ be a set of local rules with alphabet Σ . Let $\mathcal{R}_2 = \mathcal{R}_{\tau, f, g}$ and $\alpha : T \rightarrow \mathcal{R}_2$ an assignment that maps $\alpha(t) = h_t$ for all $t \in T$.

Suppose that the update rule H_{θ} is surjective (respectively pre-injective) for all $\theta \in \mathcal{R}_1^{\mathbb{Z}}$ such that there is at most one length n block of adjacent rules f in θ . Then the update rule $H_{\tau_{\alpha}}$ is surjective (respectively pre-injective).

Proof Let $B(c) = \{e \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}} \mid e^{(1)} = c^{(1)}\}$ for any $c \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ be the set of configurations that share the background track of c . For all $e \in B(c)$, rules f and g are used in the same cells, meaning there is a $\theta \in \mathcal{R}_1^{\mathbb{Z}}$ such that $H_{\tau_{\alpha}}(e)^{(2)} = H_{\theta}(e^{(2)})$ for all $e \in B(c)$.

Furthermore, because pattern $t_1 \dots t_n$ appears in τ only once, the pattern $h_{t_1} \dots h_{t_n}$ appears only once in τ_{α} . Then for any $e \in B(c)$, there is at most one length n block of cells where rule f is used, because there is at most one length n block of correct guesses for the templates in e that are also a part of the substrings $1 \dots n$ of $a1 \dots nb$, where $a \neq n$ and $b \neq 1$, in the background track. Therefore θ has at most one block of adjacent rules f .

Suppose then that for every such θ , H_{θ} is surjective. Let $c_1 \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ and $\theta \in \mathcal{R}_1^{\mathbb{Z}}$ such that $H_{\tau_{\alpha}}(e)^{(2)} = H_{\theta}(e^{(2)})$ for all $e \in B(c_1)$. Then because H_{θ} is surjective, there is $c_2 \in \Sigma^{\mathbb{Z}}$ such that $H_{\theta}(c_2) = c_1^{(2)}$. Let then $c_3 \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ such that $c_3^{(1)} = c_1^{(1)}$ and $c_3^{(2)} = c_2^{(2)}$. Then $c_3 \in B(c_1)$ and hence $H_{\tau_{\alpha}}(c_3) = c_1$. Therefore $H_{\tau_{\alpha}}$ is surjective.

Suppose then that for every such θ , H_{θ} is pre-injective. Let $c_1, c_2 \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ be asymptotic configurations with $c_1 \neq c_2$. If $c_1^{(1)} \neq c_2^{(1)}$ then clearly $H_{\tau_{\alpha}}(c_1) \neq H_{\tau_{\alpha}}(c_2)$ because the background track is static. Assume then that $c_1^{(1)} = c_2^{(1)}$, meaning $c_1^{(2)} \neq c_2^{(2)}$. Then $B(c_1) = B(c_2)$, meaning there is $\theta \in \mathcal{R}_1^{\mathbb{Z}}$ such that $H_{\tau_{\alpha}}(c_1)^{(2)} = H_{\theta}(c_1^{(2)})$ and $H_{\tau_{\alpha}}(c_2)^{(2)} = H_{\theta}(c_2^{(2)})$. Now because H_{θ} is pre-injective, $H_{\tau_{\alpha}}(c_1)^{(2)} = H_{\theta}(c_1^{(2)}) \neq H_{\theta}(c_2^{(2)}) = H_{\tau_{\alpha}}(c_2)^{(2)}$ and hence $H_{\tau_{\alpha}}(c_1) \neq H_{\tau_{\alpha}}(c_2)$. Therefore $H_{\tau_{\alpha}}$ is pre-injective. \square

Theorem 15 Let T be a set of rule templates and $\tau \in T^{\mathbb{Z}}$ be a non-recurrent rule distribution template. There exists a set of local rules \mathcal{R} and assignment $\alpha : T \rightarrow \mathcal{R}$ such that $H_{\tau_{\alpha}}$ is pre-injective, but not surjective.

Proof Let $n \geq 1$ be such that there is a length n pattern $t_1 \dots t_n$ in τ that only appears once. Let $\mathcal{R}' = \{f_n, g_n\}$. By Lemma 11 we know that for all $\theta \in \mathcal{R}'^{\mathbb{Z}}$ such that a length n block of rules f_n appears at most once, H_{θ} is pre-injective. Let $\mathcal{R} = \mathcal{R}_{\tau, f_n, g_n}$ and $\alpha : T \rightarrow \mathcal{R}$ such that $\alpha(t) = h_t$ for all $t \in T$. Then by Lemma 14, $H_{\tau_{\alpha}}$ is pre-injective.

Next, assume the pattern $t_1 \dots t_n$ is in the cells $[x + 1, x + n]$, meaning $\tau(x + i) = t_i$ when $1 \leq i \leq n$. Let then $c \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ be such that $c(x + n + 1)^{(1)} \neq 1, c(x)^{(1)} \neq n, c(x + i)^{(1)} = i$ when $1 \leq i \leq n$, and $c^{(2)}$ is a configuration with no pre-image as in the proof of Lemma 10, for a rule distribution where the n block of f -cells is in the range $[x + 1, x + n]$. Now rule f_n is used in cells $x + 1, \dots, x + n$ and rule g_n is used in x . By

Lemma 10, c then has no pre-image and therefore H_{τ_α} is not surjective. \square

Theorem 16 *Let T be a set of rule templates and $\tau \in T^{\mathbb{Z}}$ be a non-recurrent rule distribution template. There exists a set of local rules \mathcal{R} and assignment $\alpha : T \rightarrow \mathcal{R}$ such that H_{τ_α} is surjective, but not pre-injective.*

Proof Let $n \geq 1$ be such that there is a length n pattern $t_1 \dots t_n$ in τ that only appears once. Let $\mathcal{R}' = \{\gamma_n, \delta_n\}$. By Lemma 13 we know that for all $\theta \in \mathcal{R}'^{\mathbb{Z}}$ such that a length n block of rules γ_n appears at most once, H_θ is surjective. Let $\mathcal{R} = \mathcal{R}_{\tau, \gamma_n, \delta_n}$ and $\alpha : T \rightarrow \mathcal{R}$ such that $\alpha(t) = h_t$ for all $t \in T$. Then by Lemma 14, H_{τ_α} is surjective.

Next, assume the pattern $t_1 \dots t_n$ is in the cells $[x + 1, x + n]$, meaning $\tau(x + i) = t_i$ when $1 \leq i \leq n$. Let $c, e \in (\mathbb{Z}_n \times \Sigma)^{\mathbb{Z}}$ be such that $c^{(1)} = e^{(1)}$, $c(x + n + 1)^{(1)} \neq 1$, $c(x)^{(1)} \neq n$, $c(x + i)^{(1)} = i$ when $1 \leq i \leq n$ and $c^{(2)}, e^{(2)}$ are asymptotic but different configurations with the same image as in Lemma 12, when the n block of γ -cells is in the range $[x + 1, x + n]$. Now rule γ_n is used in cells $x + 1, \dots, x + n$ and rule δ_n is used in x and $x + n + 1$. Then by Lemma 12, c and e are asymptotic differing configurations with the same image, meaning H_{τ_α} is not pre-injective. \square

A similar reduction to the preceding can be achieved by using a technique found in Salo (2014). Using this method, the assignment can be defined with a fixed number of states, not depending on the number of templates.

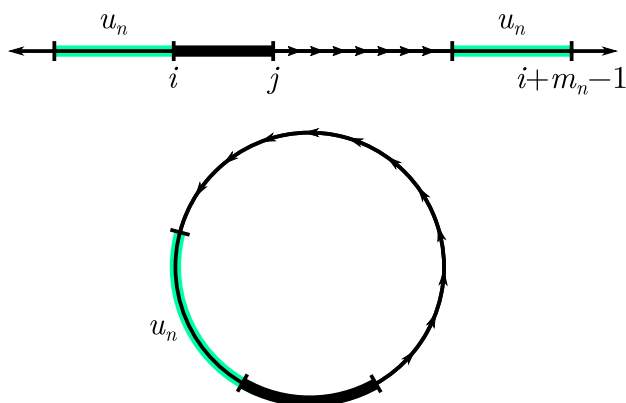


Fig. 13 The wrapping of θ (above) to ψ_n (below). The arrows indicate the direction of the wrapping

6 Surjectivity

Surjectivity is the property of a cellular automaton that if its global update function is injective, then it is surjective. Surjectivity is implied by the Garden of Eden theorem, so it is known to hold for regular CA over integer grids.

Furthermore, then we also know that surjectivity holds for NUCA with a recurrent rule distribution. In this section we show a property of a rule distribution that implies surjectivity, and show that every distribution asymptotic to a recurrent distribution has said property.

First we define NUCA over a finite space \mathbb{Z}_n for some $n \in \mathbb{Z}_+$. The definition is analogous to a typical NUCA. Intuitively, if a normal NUCA is thought of as acting on a bi-infinite "tape", a NUCA over \mathbb{Z}_n acts on a finite, circular tape.

Definition 16 Let $n \in \mathbb{Z}_+$, \mathcal{R} a rule set over state set Σ and with neighbourhood $N = (n_1, \dots, n_m)$, where $n_1, \dots, n_m \in \mathbb{Z}_n$. Let $\psi \in \mathcal{R}^{\mathbb{Z}_n}$ be a rule distribution over the finite set \mathbb{Z}_n . The tuple $A = (\Sigma, \mathbb{Z}_n, N, \mathcal{R}, \psi)$ is a *non-uniform CA over finite space*, and its *global update function* $H_\psi : \Sigma^{\mathbb{Z}_n} \rightarrow \Sigma^{\mathbb{Z}_n}$ is the one that maps $c \in \Sigma^{\mathbb{Z}_n}$ such that

$$H_\psi(c)(x) = \psi(x)((x + n_1 \bmod n), \dots, (x + n_m \bmod n)),$$

for all $x \in \mathbb{Z}_n$. If A is clear from context, it's usually referred to by its global update rule.

For any $k \in \mathbb{Z}$, we denote $(-\infty, k) = \{x \in \mathbb{Z} \mid x < k\}$ and $(k, \infty) = \{x \in \mathbb{Z} \mid x > k\}$.

Lemma 17 *Let $\phi \in \mathcal{R}^{\mathbb{Z}}$ be a recurrent rule distribution. Let $i, j \in \mathbb{Z}$, $i \leq j$, and $u = \phi|_{(-\infty, i)}$, $v = \phi|_{[i, j]}$, $w = \phi|_{(j, \infty)}$. Either every finite suffix of u is a subword of w or every finite prefix of w is a subword of u .*

Proof Suppose x is a suffix of u and y is a prefix of w such that x is not a subword of w and y is not a subword of u . Then the word xvy cannot appear infinitely many times in ϕ ; if it did, either x would be a subword of w or y a subword of u . Then ϕ isn't recurrent, which is a contradiction. Therefore either x must be a subword of w or y must be a subword of u . \square

We show that for any distribution θ with the property shown for recurrent distributions in Lemma 17, the NUCA H_θ will be surjective.

Lemma 18 *Let $\theta \in \mathcal{R}^{\mathbb{Z}}$ be a rule distribution. Let $i, j \in \mathbb{Z}$ such that $i, j \in \mathbb{Z}$, $i \leq j$, $u = \theta|_{(-\infty, i)}$, $w = \theta|_{(j, \infty)}$, and assume that either every finite suffix of u is a subword of w*

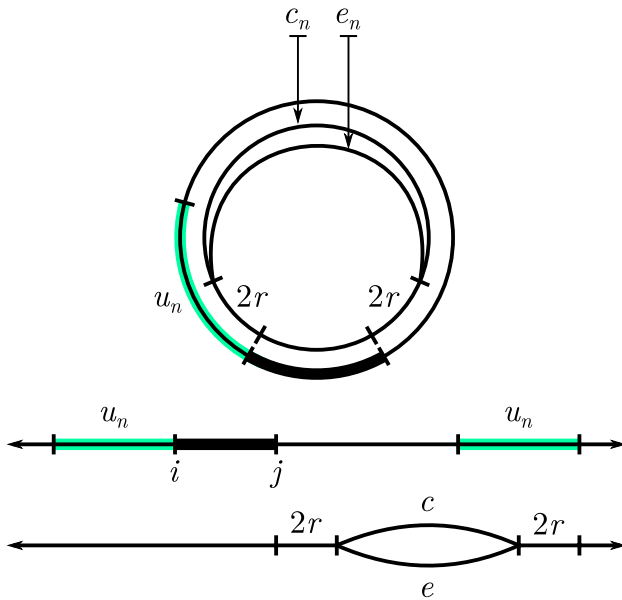


Fig. 14 The unwrapping of c_n and e_n into c and e in Case 2 of the proof of Lemma 13

or every finite prefix of w is a subword of u . Then if H_θ is injective, it is surjective.

Proof Assume H_θ is not surjective. Let Σ be the state set of rules in \mathcal{R} and assume that all rules are at most radius r . Assume that every suffix of u is a subword of w . The other case is identical.

For any $n \in \mathbb{Z}_+$, let $u_n = \theta_{|[i-n, i-1]}$ be the length n suffix of $\theta_{|(\infty, i)}$. Let m_n be the length of the segment from i to the rightmost cell of the leftmost copy of u_n to the right of j . Let then $\psi_n \in \mathcal{R}^{\mathbb{Z}^{m_n}}$ be such that $\psi_n(x \bmod m_n) = \theta(x)$ for all $x \in [i, i + m_n - 1]$. We can think of ψ_n as the segment of θ ranging from u_n to a copy of u_n , wrapped around in a circle with the two copies overlapping each other. This is illustrated in Fig. 13.

Because H_θ is not surjective, by Lemma 1, for large enough n the function H_{ψ_n} is not surjective. Since $\Sigma^{\mathbb{Z}^{m_n}}$ is finite, H_{ψ_n} is not injective. Let then $c_n, e_n \in \Sigma^{\mathbb{Z}^{m_n}}$ be configurations such that $c_n \neq e_n$ and $H_{\psi_n}(c_n) = H_{\psi_n}(e_n)$. Now there are two cases.

Case 1: For infinitely many n , the c_n and e_n differ within $2r$ of the segment $[i, j]$, that is, $\text{diff}(c_n, e_n) \cap [i - 2r, j + 2r] \neq \emptyset$. Let then $c'_n, e'_n \in \Sigma^{\mathbb{Z}}$ be configurations such that $c'_n(x) = c_n(x \bmod m_n)$ and $e'_n(x) = e_n(x \bmod m_n)$ for all $x \in [i - n, i + m_n - 1]$ and $c'_n(y) = e'_n(y)$ for all $y \notin [i - n, i + m_n - 1]$. The configurations c'_n and e'_n can be thought of as c_n and e_n "unwrapped" and "embedded" into some configuration over \mathbb{Z} .

Let then $(c'_{n_k}, e'_{n_k})_k$ be a sequence of pairs of these unwrapped configurations, where the indices n_k are the ones where c_{n_k} and e_{n_k} differ within $2r$ of $[i, j]$. By compactness, this sequence has a converging subsequence with a limit (c, e) . Now $c \neq e$, because $\text{diff}(c'_{n_k}, e'_{n_k}) \cap [i - 2r, j + 2r] \neq \emptyset$ for all k . In addition, because $H_{\psi_{n_k}}(c_{n_k}) = H_{\psi_{n_k}}(e_{n_k})$ for all k , it holds that for any finite domain $D \subset \mathbb{Z}$ there is $m \in \mathbb{Z}_+$ such that $H_\theta(c'_{n_k})(D) = H_\theta(e'_{n_k})(D)$ for all $k \geq m$. Therefore $H_\theta(c) = H_\theta(e)$, meaning H_θ is not injective.

Case 2: For all large enough n , c_n and e_n are identical within $2r$ of $[i, j]$. Let $c, e \in \Sigma^{\mathbb{Z}}$ such that for some such n , $c(x) = c_n(x \bmod m_n)$ and $e(x) = e_n(x \bmod m_n)$ for all $x \in [j + 1, i + m_n - 1]$ and $c(y) = e(y)$ for all $y \notin [j + 1, i + m_n - 1]$. This is illustrated in Fig. 14.

Clearly $c \neq e$, because c_n and e_n differ somewhere in the segment $[j + 1, i + m_n - 1]$. For any cell x that is at least r cells away from $\text{diff}(c, e)$, the neighbourhood of x is identical in c and e , hence $H_\theta(c)(x) = H_\theta(e)(x)$. For any cell y that is within r cells of $\text{diff}(c, e)$, its neighbourhood is within $2r$ of $\text{diff}(c, e)$. Because $H_{\psi_n}(c_n) = H_{\psi_n}(e_n)$, then $H_\theta(c)(y) = H_\theta(e)(y)$. Therefore $H_\theta(c) = H_\theta(e)$, meaning H_θ is not injective.

Hence in either case, H_θ is not injective. Therefore if H_θ is injective, it is surjective. \square

Theorem 19 Let $\theta \in \mathcal{R}^{\mathbb{Z}}$ be asymptotic to a recurrent rule distribution $\phi \in \mathcal{R}^{\mathbb{Z}}$. If H_θ is injective, it is surjective.

Proof Let $i, j \in \mathbb{Z}$, $i \leq j$ be such that $\text{diff}(\theta, \phi) \subseteq [i, j]$ and let $u = \phi_{|(i, \infty)}$ and $w = \theta_{|(j, \infty)}$. By Lemma 17, either every finite suffix of u is a subword of w or every finite prefix of w is a subword of u . Then because ϕ and θ are identical outside of $[i, j]$, by Lemma 13, if H_θ is injective, it is surjective. \square

7 Conclusions

We find that the Garden of Eden theorem holds for NUCA if the local rule distribution is uniformly recurrent. In the 1-dimensional case we find that every assignment to a given rule template defines a NUCA that satisfies either direction of the Garden of Eden theorem, if and only if the template is recurrent. Finally we find that all rule distributions asymptotic to a recurrent distribution are surjective.

The Garden of Eden theorem for NUCA should still be examined in other groups. As for surjectivity, we have shown a property of a template that guarantees surjectivity, but know nothing about the converse. It may be useful to examine the complement of the underlying property which

gives us surjectivity, and see whether this guarantees the existence of non-surjective assignments.

Author contributions K. Paturi (née P. Paturi) wrote the main manuscript text and J. Kari provided large portions of the proofs of theorems 3, 4 and 5. Both authors reviewed the manuscript.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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